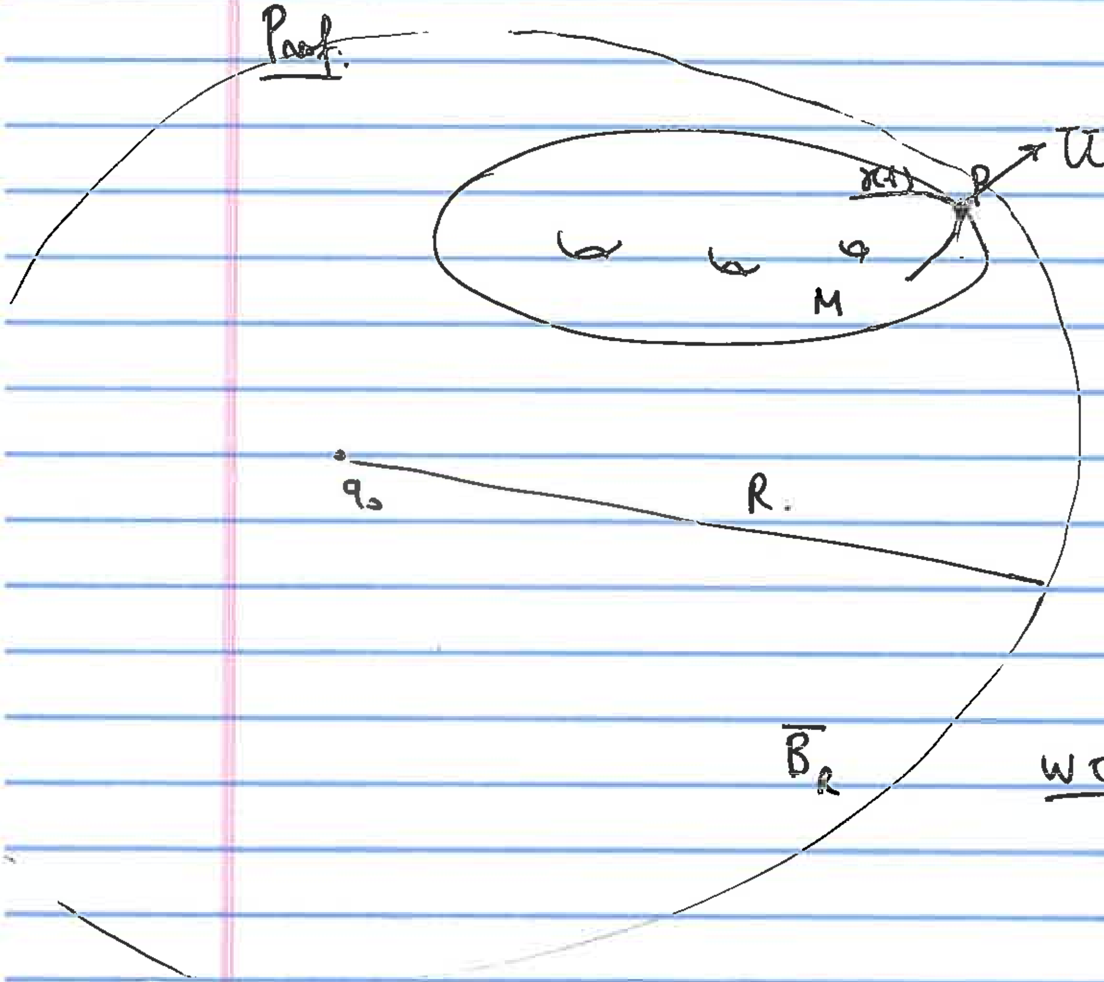
3.5 Only one prop we will do. (No HW from 3.5)

Prop: Every compact surface M in $\mathbb{R}^3$ has a point of positive Gaussian curvature

In fact,

If $\exists\, q_0,\ R>0$ s.t. $M \subseteq B_R(q_0)$, then $\exists\, p \in M$ $K(p) \geq \frac{1}{R^2} > 0$.

Proof.



Let $g = \|q_0 - x\|^2$ $x \in M$.

M compact

g is continuous in x

$\Rightarrow \exists$ must exist a max for g.

Let be the furthest pt of M from q_0.

WTS $K(p) \geq \frac{1}{R^2} > 0$

Let $\gamma(t)$ be a curve in M s.t. $\gamma(0) = p$; $|\gamma'| = 1$

$h(t) = \|q_0 - \gamma(t)\|^2$ $\qquad$ h has a max at $t=0$

$h' = 2T \cdot (\gamma - q_0)$ $\qquad$ $h'(0) = 0$

$h'' = 2[\kappa_\gamma N_\gamma \cdot (\gamma - q_0) + 1]$ $\qquad$ $h''(0) \leq 0$

$$h'(0) = 0 = 2\overset{\gamma'(0)}{\underset{\sim}{T}}(0) \cdot (\gamma(0) - q_0) = 0$$

$$\underbrace{k_\gamma(0) \cdot N_\gamma(0) \cdot (p - q_0)}_{< 0} + 1 \leq 0$$

$$|N_\gamma(0) \cdot (p - q_0)| \leq |N_\gamma||p - q_0| \leq 1 \cdot R$$

$$\Rightarrow \quad k_\gamma(0) \geq \frac{1}{R}$$

$$N_\gamma(0) \cdot (p - q_0) < 0$$

If one takes U outward normal at p
$\left(U = \frac{p - q_0}{|p - q_0|} \right)$

$$\Rightarrow \quad k_n = k \cdot \cos\theta \leq -\frac{1}{R}.$$

$$\theta = \underbrace{\measuredangle(N_\gamma, U)}_{\pi}$$

since p is the furthest pt of M to q_0

all normal curvatures $\leq -\frac{1}{R}$.

$$k_1, k_2 \leq -\frac{1}{R}.$$

$$K(p) = k_1 k_2 \geq \frac{1}{R^2} > 0.$$

Corollary: 1) $\nexists$ no compact surface in $\mathbb{R}^3$ with $K \leq 0$
2) $\nexists$ no compact $\underbrace{\text{minimal surface in } \mathbb{R}^3}$.

$$\frac{k_1 + k_2}{2} = H \equiv 0$$

$$k_1 k_2 \leq 0$$

CHAP V

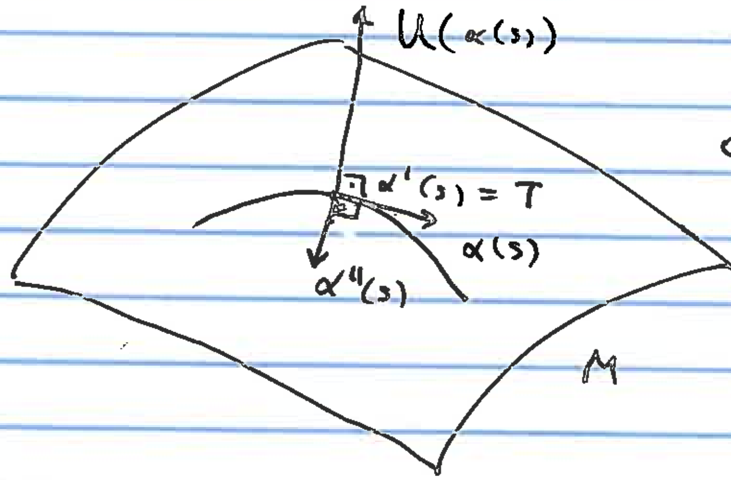
5.1 Geodesics

Let M be a regular surface,
Let $\alpha(s) : I \to M$ be C^∞ s.t. $|\alpha'| \equiv 1$.

$\Downarrow$

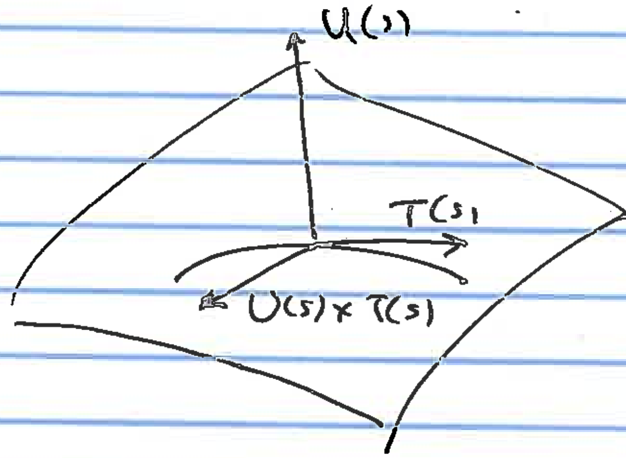
$\alpha' \cdot \alpha'' \equiv 0$

$0 = \alpha'(s) \cdot U(s)$



$$\alpha''(s) = \kappa_\alpha(s) \cdot N_\alpha(s)$$
$$\alpha'(s) = T_\alpha(s)$$
$$U(s) = U(\alpha(s)).$$

$\{U(s), T(s), U(s) \times T(s)\}$ orthonormal basis of $\mathbb{R}^3$ at $\alpha(s)$



Obs $U \times T \perp U$
$\Rightarrow U \times T$ is tangent to M

Recall $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ o.n. $\Rightarrow$ $\vec{w} = (\vec{w}\cdot\vec{u}_1)\vec{u}_1 + (\vec{w}\cdot\vec{u}_2)\vec{u}_2 + (\vec{w}\cdot\vec{u}_3)\vec{u}_3$

$$\alpha''(s) = \underbrace{(\alpha''\cdot T)}_{0} T + \underbrace{(\alpha''\cdot U\times T)}_{\text{Want } k_g} U\times T + \underbrace{(\alpha''\cdot U)}_{k_n.} U$$

$|\alpha'| = 1$

$\Rightarrow \alpha'\cdot\alpha'' = 0$

Recall

$$\alpha''\cdot U = S_p(\alpha')\cdot\alpha' = II_p(\alpha') = k_n(\alpha')$$

$|\alpha'| = 1$, $|\alpha''| = k_\alpha$. curve curvature.

Defn ① If $|\alpha'(s)| = 1$, then

$$k_g = \big(U(\alpha(s))\times\alpha'(s)\big)\cdot\alpha''(s)$$ is called the <u>geodesic curvature</u> of α in M.

②

$$\alpha'' = \underbrace{\underbrace{(\alpha''\cdot U\times T)}_{k_g}\cdot U\times T}_{\alpha''_{tan}} + \underbrace{\underbrace{(\alpha''\cdot U)}_{k_n}\cdot U}_{\alpha''_{normal}}$$

Obs:

$U\times T \perp U$

<u>$U\times T$ tangent to M</u>

referring to tangent to M

" " normal to M.

$$\alpha'' = \alpha''_{tan} + \alpha''_{normal}.$$

→ For $\alpha(t)$ not necessarily parametrized wrt arclength:

Recall Chap I

$$\alpha' = vT \qquad v = \text{speed}$$

$$\alpha'' = v'T + vT' = v'T + v^2 \kappa N$$

curve normal

$$\alpha'' = (\alpha'' \cdot T)\,T + (\alpha'' \cdot U \times T)\,U \times T + (\alpha'' \cdot U)\,U$$

$$= \underbrace{v'T + v^2 \kappa_g (U \times T)}_{\alpha''_{tan}} + \underbrace{v^2 k_n U}_{\alpha''_{normal}}$$

Obs Changing speed doesn't change

κ curve curvature
κ_g geodesic "
k_n normal "

Defn A curve $\alpha : I \to M$ is called a geodesic if $\alpha''_{tan} \equiv 0$.

Lemma: $\underbrace{\alpha \text{ geodesic}}_{\alpha''_{tan} \equiv 0} \iff \begin{cases} v \equiv \text{constant (speed) and} \\ \kappa_g \equiv 0 \end{cases}$

(unless α = one pt, $v \equiv 0$.)

Proof, ⇐ obvious

⇒ recall T & $U \times T$ are linearly independent

$$v'T + v^2 \kappa_g (U \times T) = 0$$

$$\Rightarrow v' = 0 \text{ and}$$

$$v^2 \kappa_g \equiv 0 .$$