3.4 Continue

Equations of Compatibility

① $\Psi_{uu} = \Gamma_{uu}^{u}\Psi_u + \Gamma_{uu}^{v}\Psi_v + l\,\vec{U}$

② $\Psi_{uv} = \Gamma_{uv}^{u}\Psi_u + \Gamma_{uv}^{v}\Psi_v + m\,\vec{U}$

③ $\Psi_{vv} = \Gamma_{vv}^{u}\Psi_u + \Gamma_{vv}^{v}\Psi_v + n\,\vec{U}$

$-S_p(\Psi_u) = U_u = -a\Psi_u - b\Psi_v$

$-S_p(\Psi_v) = U_v = -c\Psi_u - d\Psi_v$

$[S_p] = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$

$\Psi_{uuv} = (\Psi_{uu})_v = (\Psi_{uv})_u = \Psi_{uvu}$ since $\Psi \in C^3$

$\Rightarrow \left(\Gamma_{uu}^{u}\Psi_u + \Gamma_{uu}^{v}\Psi_v + l\,\vec{U}\right)_v$

$= \left(\Gamma_{uv}^{u}\Psi_u + \Gamma_{uv}^{v}\Psi_v + m\,\vec{U}\right)_u$

use ② above, use ③

$(\Gamma_{uu}^{u})_v \cdot \Psi_u + \Gamma_{uu}^{u}\Psi_{uv} + (\Gamma_{uu}^{v})_v \cdot \Psi_v + \Gamma_{uu}^{v}\Psi_{vv} + \cdots$

$+ l_v\vec{U} + l\,U_v = (\Gamma_{uv}^{u})_u \cdot \Psi_u + \Gamma_{uv}^{u}\Psi_{uu} + (\Gamma_{uv}^{v})_u \cdot \Psi_v +$

use ①

$+ \Gamma_{uv}^{v} \cdot \Psi_{vu} + m_u\vec{U} + m \cdot \vec{U}_u$

use ②

$\underbrace{}$ above

Plug in ①, ②, ③ into equation, and write it as

$0 = c_1\Psi_u + c_2\Psi_v + c_3\vec{U}.$

$$0 = \psi_{uuv} - \psi_{uvu} = 0$$

$$0 = \psi_u \Big[(\Gamma^u_{uu})_v + \cancel{\Gamma^u_{uu}\Gamma^u_{uv}} + \Gamma^u_{vv}\cdot\Gamma^v_{uu} - \ell e +$$
$$-(\Gamma^u_{uv})_u - \cancel{\Gamma^u_{uv}\Gamma^u_{uu}} - \Gamma^v_{uv}\Gamma^u_{vu} + am \Big]$$
$$+ \psi_v \Big[\Gamma^u_{uu}\cdot\Gamma^v_{uv} + (\Gamma^v_{uu})_v + \Gamma^v_{vv}\Gamma^v_{uu} - \ell d +$$
$$-\Gamma^u_{uv}\Gamma^v_{uu} - (\Gamma^v_{uv})_u - \Gamma^v_{uv}\Gamma^v_{vu} + bm \Big]$$
$$+ U \Big[\Gamma^u_{uu} m + \ell_v + n\Gamma^v_{uu} - \ell\Gamma^u_{uv} - \Gamma^v_{uv} m - m_u$$

ψ_u, ψ_v, U lin independent

$\Rightarrow$ each coeff of ψ_u, ψ_v, U must be 0

Coeff ψ_u is 0:

(*1)
$$(\Gamma^u_{uu})_v - (\Gamma^u_{uv})_u + \Gamma^v_{uu}\Gamma^u_{vv} - \Gamma^v_{uv}\Gamma^u_{uv} = -am + \ell e$$
$$= -FK.$$
next page

Coff. ψ_v is 0:

(*2)
$$(\Gamma^v_{uu})_v - (\Gamma^v_{uv})_u + \Gamma^u_{uu}\Gamma^v_{uv} + \Gamma^v_{uu}\Gamma^v_{vv} - \Gamma^u_{uv}\Gamma^v_{uu} - \Gamma^v_{uv}\Gamma^v_{vu}$$
$$= -bm + d\ell = EK.$$
see next page

Page 2 Since $E \neq 0$

$$K = \frac{1}{E}\Big[(\Gamma^v_{uu})_v - (\Gamma^v_{uv})_u + \Gamma^u_{uu}\Gamma^v_{uv} + \Gamma^v_{uu}\Gamma^v_{vv} - \Gamma^u_{uv}\Gamma^v_{uu} - \Gamma^v_{uv}\Gamma^v_{vu} \Big]$$

Prop 1 Christoffel symbols can be calculated from E, F, G.

Prop 2 K can be calculated from Γ^i_{jk} & E, F, G

$\Longrightarrow$ K is calculable from E, F, G.

This is the proof of Theorema Egregium.

Missing: $-bm + dl = EK$?

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{[S_p]^T} = \underbrace{\begin{pmatrix} l & m \\ m & n \end{pmatrix}}_{\mathrm{II}} \underbrace{\begin{pmatrix} +G & -F \\ -F & +E \end{pmatrix} \frac{1}{\Delta}}_{\mathrm{I}^{-1}} \qquad \Delta = EG - F^2$$

$$b = \frac{-Fl + mE}{\Delta}.$$

$$d = \frac{-Fm + nE}{\Delta}.$$

$$-bm + dl = +\left(\frac{Fl - mE}{\Delta}\right) m + \left(\frac{-Fm + nE}{\Delta}\right) l$$

$$= \frac{1}{\Delta}\left(\cancel{Flm} - m^2 E - \cancel{Flm} + nlE\right) = \frac{nl - m^2}{EG - F^2} \cdot E = KE.$$