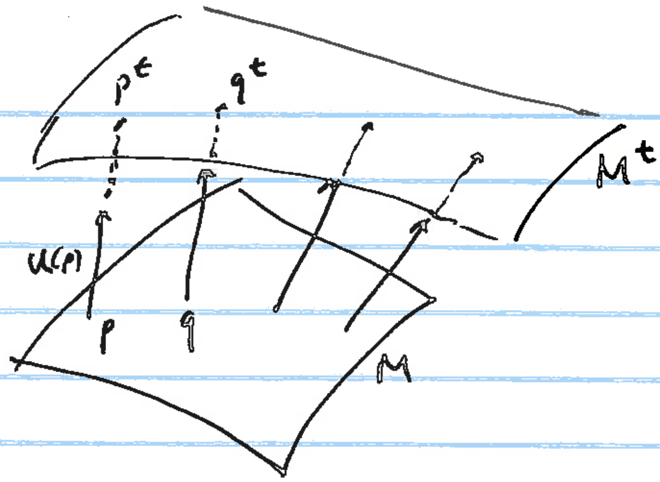
Parallel Surfaces

Ex 3.2.6



$$p^t = p + tU(p)$$

Let $\Psi(u,v)$ be a param. of M, about p.

$$\Psi^t = \Psi + tU \qquad U \text{ normal of } M.$$

at p^t

$$\Psi^t_u = \Psi_u + tU_u = \Psi_u - tS_p^\circ(\Psi_u)$$
$$\Psi^t_v = \Psi_v + tU_v = \Psi_v - tS_p^\circ(\Psi_v)$$

$$\Psi^t_u \times \Psi^t_v = \left(\Psi_u - tS_p^\circ(\Psi_u)\right) \times \left(\Psi_v - tS_p^\circ(\Psi_v)\right)$$

$$= \Psi_u \times \Psi_v - t\left(S_p^\circ(\Psi_u) \times \Psi_v + \Psi_u \times S_p^\circ(\Psi_v)\right) +$$

$$+ t^2\left(S_p^\circ(\Psi_u) \times S_p^\circ(\Psi_v)\right)$$

Pn ⊛ **Recall** 1) $S_p(v) \times S_p(w) = K(p)(v \times w)$
2) $S_p(v) \times w + v \times S_p(w) = 2H(p)(v \times w)$

$$\underbrace{\Psi^t_u \times \Psi^t_v}_{\text{at } p^t} = \left(\Psi_u \times \Psi_v\right)\Big(\underbrace{1 - t\,2H(p) + t^2 K(p)}_{\Delta(t)}\Big) \quad \text{at } p.$$

$$\psi_u^t \times \psi_v^t = (\psi_u \times \psi_v)\, \Delta(t)$$

recall $\quad K(p) = k_1 k_2$

$\qquad\qquad H(p) = (k_1 + k_2)/2.$

$$(\lambda - k_1)(\lambda - k_2) = \lambda^2 - (k_1 + k_2)\lambda + k_1 k_2$$
$$= \lambda^2 - 2H\lambda + K = 0$$

soln's $\lambda = k_1$

$\lambda = k_2$

$\Delta(t) = 1 - 2tH + t^2 K = (1 - tk_1)(1 - tk_2)$ at p.

$\psi_u \times \psi_v \neq 0$; but $\Delta(t) = 0 \iff t = \frac{1}{k_1}$ or $\frac{1}{k_2}$

$\sigma(t) = \operatorname{sign} \Delta(t)$ when $\Delta(t) \neq 0$

$$\underset{\text{at } p.}{U} = \frac{\psi_u \times \psi_v}{|\psi_u \times \psi_v|} = \sigma \frac{\psi_u^t \times \psi_v^t}{|\psi_u^t \times \psi_v^t|} = \sigma U^t = \pm \underset{\text{at } p^t}{U^t}$$

$$U^t = \sigma(t) U$$

M^t is singular when $\Delta(t) = 0$;

Caution k_1, k_2 change with p.

M^t have singularities at different t values for varying p: $t = \underbrace{\frac{1}{k_i}}_{\text{varies}}(p)$

$\{\Psi_u, \Psi_v\}$ basis for T_pM

$\{\Psi_u^t, \Psi_v^t\}$ basis for $T_{p^t}M^t$

$\sigma = \pm 1$ when $\Delta \neq 0$.

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$$S_{p^t}^t(\Psi_u^t) = -\frac{\partial}{\partial u} U^t = -\frac{\partial}{\partial u}\sigma U$$
$$= \sigma\left(-\frac{\partial}{\partial u} U\right) = \sigma S_p^\circ(\Psi_u)$$

Recall Prop ⊛ (1)

$$K^t(p^t)(\Psi_u^t \times \Psi_v^t) = S_{p^t}^t(\Psi_u^t) \times S_{p^t}^t(\Psi_v^t)$$
$$= \sigma S_p^\circ(\Psi_u) \times \sigma S_p^\circ(\Psi_v) = \sigma^2 S_p^\circ(\Psi_u) \times S_p^\circ(\Psi_v)$$
$$= K(p)\,\Psi_u \times \Psi_v$$

Recall $\Psi_u^t \times \Psi_v^t = \Delta(t)\left[\Psi_u \times \Psi_v\right]$

$$\boxed{K^t(p^t) = \frac{K(p)}{\Delta(t)(p)}}$$

Prop ⊛ 2:

$$2H^t(p^t)(\Psi_u^t \times \Psi_v^t) = S_{p^t}^t(\Psi_u^t) \times \Psi_v^t + \Psi_u^t \times S_{p^t}^t(\Psi_v^t)$$

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$$= \sigma S_p^\circ(\Psi_u) \times \underbrace{\left(\Psi_v - t S_p^\circ(\Psi_v)\right)}_{\Psi_v^t \text{ page ①}} + \underbrace{\left(\Psi_u - t S_p^\circ(\Psi_u)\right)}_{\Psi_u^t} \times \sigma S_p^\circ(\Psi_v)$$

$$= \sigma\left[S_p^\circ(\Psi_u) \times \Psi_v + \Psi_u \times S_p^\circ(\Psi_v)\right] +$$
$$-\sigma t\left[S_p^\circ(\Psi_u) \times S_p^\circ(\Psi_v) + S_p^\circ(\Psi_u) \times S_p^\circ(\Psi_v)\right]$$
$$= \sigma\left[2H(p) - 2K(p)t\right](\Psi_u \times \Psi_v)$$

In Summary
We obtained

$$2H^t(p^t)\,\underbrace{(\Psi_u^t \times \Psi_v^t)}_{\Delta(t)\,\Psi_u\times\Psi_v} = 2\sigma\,[H-Kt]\,\Psi_u\times\Psi_v \quad (\text{at } p)$$

at p^t

$$\boxed{H^t_{(p^t)} = \sigma\,\frac{H-Kt}{\Delta}\,(p)}$$

What about principal curvatures?
we claim:

$\sigma>0$

$$\left.\begin{aligned} k_1^t &= \frac{k_1}{1-k_1t} \\ k_2^t &= \frac{k_2}{1-k_2t} \end{aligned}\right\}$$

we claim that these are the only solns of
$0 = \lambda^2 - 2H^t\lambda + K^t$

$$k_1^t k_2^t = \frac{k_1}{1-k_1t}\cdot\frac{k_2}{1-k_2t} = \frac{k_1k_2}{(1-k_1t)(1-k_2t)} = \frac{K}{\Delta} = K^t$$

$$\frac{k_1^t+k_2^t}{2} = \frac{1}{2}\left(\frac{k_1}{1-tk_1}+\frac{k_2}{1-tk_2}\right)$$

$$= \frac{1}{2}\left(\frac{k_1 - tk_1k_2 + k_2 - tk_1k_2}{(1-tk_1)(1-tk_2)}\right)$$

$$= \frac{1}{2\Delta}\Big(\underbrace{k_1+k_2}_{2H} - 2t\underbrace{k_1k_2}_{K}\Big) = \frac{H-tK}{\Delta} = H^t$$

Since a quadratic has only 2 solutions, these are the roots.