MATH 4500
Take-home MIDTERM
Due 10:30 am, Monday, October 9, 2017

You are allowed to use any theorem, proof, calculation or example done in class or in the sections of the textbook we covered so far, as long as you clearly indicate what you are using. If you want to use a computer to check some graphs of curves or calculations, it is fine. You still need to show details of your calculations and proofs written by you. For a take-home test, students are allowed to use their notes, textbook, to ask your instructor questions/hints in class. Discussing the test questions with other students is **not** allowed. Using another person's solutions is **not** allowed. Using other textbooks and online resources are **not** allowed.

Since this is take home-exam, your solutions should be written legibly, showing details and steps, and your answers should be simplified as much as possible. Please include a cover page and start with a new page with each question. Doing these will give me space to write grades and comments. You can write on both sides of the paper if you would like. Either staple your pages together or write your name on each page.

There are 7 questions in this test.
First question is 10 points and the rest are 15 points each, for a total of 100 points.

1. For the curve $\alpha(t) = (6t^2, 3t, 4\sqrt{2}\,t^{\frac{3}{2}})$, $t \geq 0$:
   a. Calculate the arclength function, starting at $t = 0$,
   b. Reparamatrize α with respect to arclength, and
   c. Find its length for $1 \leq t \leq 4$.

2. Calculate the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$, the curvature and the torsion of the curve $\alpha(t) = \left(-2t, t^2, \frac{1}{3}t^3\right)$, $t \in \mathbb{R}$.

3. From the textbook: Do the exercises 1.3.12 and 1.3.13, page 21.

4. From the textbook: Do the exercises 1.3.24, page 25.

5. From the textbook: Read the definition of the plane evolute from Example 1.4.11, page 31, and do the exercise 1.4.14, page 33. Take $a > 0$ and $b > 0$.

PLEASE TURN THE PAGE OVER FOR QUESTIONS 6 and 7.

6. Let $\alpha(s) : I \to \mathbb{R}^3$ be a smooth (C^∞) curve in $\mathbb{R}^3$ such that $\kappa_\alpha > 0$, and $\|\alpha'(s)\| = 1$. Let $\mathbf{T}_\alpha, \mathbf{N}_\alpha, \mathbf{B}_\alpha, \kappa_\alpha, \tau_\alpha$, and v_α denote the unit tangent vector, the principal normal, the binormal, the curvature, the torsion and the speed of α, respectively.

Let $\beta(s) = \mathbf{T}_\alpha(s)$, which is called the Tangent Spherical Image of α or Tangent Indicatrix (Tantrix) of α.

Calculate the speed, the unit tangent vector, the principal normal, the binormal, the curvature, and the torsion of β (which are $v_\beta, \mathbf{T}_\beta, \mathbf{N}_\beta, \mathbf{B}_\beta, \kappa_\beta$, and τ_β, respectively) in terms of $\mathbf{T}_\alpha, \mathbf{N}_\alpha, \mathbf{B}_\alpha, \kappa_\alpha, \tau_\alpha$.

CAUTION: $v_\alpha = 1$, but s is not necessarily an arclength parameter of β.

HINTS: First prove that $v_\beta = \kappa_\alpha, \mathbf{T}_\beta = \mathbf{N}_\alpha$ and $\kappa_\beta(s) = \sqrt{1 + (\frac{\tau_\alpha}{\kappa_\alpha})^2}$. Then prove and provide simplified formulas for $\mathbf{N}_\beta, \mathbf{B}_\beta$, and prove that $\tau_\beta = \frac{(\frac{\tau_\alpha}{\kappa_\alpha})'}{\kappa_\alpha\left(1+(\frac{\tau_\alpha}{\kappa_\alpha})^2\right)}$.

7. Let $\alpha(s) : [0, L] \to \mathbb{R}^2$ be a smooth (C^∞) curve with $\|\alpha'(s)\| = 1$ and curvature $\kappa_\alpha(s) > 0$ so that N_α exists on all of $[0, L]$. A curve β is called parallel to α if it is defined by

$$\beta(s) = \alpha(s) - rN_\alpha(s)$$

where $r \in \mathbb{R}$ is a chosen constant and $N_\alpha(s)$ is the normal of α. Different choices of r give different parallels to α.

a. Prove that the curvature κ_β of β satisfies that

$$\kappa_\beta(s) = \left| \frac{\kappa_\alpha(s)}{1 + r\kappa_\alpha(s)} \right| \text{ when } \beta \text{ is regular at } \beta(s).$$

b. Give an example of regular curve $\alpha(s)$, a point $\alpha(s_0)$ on it, an r value, but β is NOT regular at $\beta(s_0)$.

c. Prove that (The length of β) $= 2\pi r +$ (The length of α), if α is a simple closed strictly convex smooth plane curve and $r > -(\max\kappa_\alpha)^{-1}$.

A simple closed smooth curve $\alpha(s) : [0, L] \to \mathbb{R}^2$ is called **strictly convex** if its curvature $\kappa_\alpha > 0$ on all of $[0, L]$, and the derivatives of all orders k match at the initial = terminal point: $\alpha^{(k)}(0) = \alpha^{(k)}(L)$. For example, circles and ellipses are strictly convex. You may assume that the rotation index of a simple closed curve is 1 if oriented counterclockwise, without proof. However, provide an explanantion why the signed curvature and standard curvature are the same for a strictly convex counterclockwise oriented closed curve, if you use such a fact.

#1 $\alpha(t) = (6t^2, 3t, 4\sqrt{2}t^{3/2})$

$$\alpha'(t) = (12t, 3, 6\sqrt{2}t^{1/2})$$

$$|\alpha'(t)| = \sqrt{144t^2 + 9 + 72t}$$
$$= 3\sqrt{16t^2 + 8t + 1}$$
$$= 3(4t+1)$$

$$s(t) = \int_0^t 3(4u+1)\,du = 3\cdot(2u^2 + u)\Big|_0^t$$
$$= 3(2t^2 + t) = 6t^2 + 3t.$$

$$s = 6t^2 + 3t \implies 6t^2 + 3t - s = 0$$

$$t = \frac{-3 \pm \sqrt{9 + 24s}}{12}$$

b) $$\alpha(t(s)) = \left(6\cdot\left(\frac{\sqrt{9+24s} - 3}{12}\right)^2, 3\cdot\left(\frac{\sqrt{9+24s} - 3}{12}\right), 4\sqrt{2}\left(\qquad\right)^{3/2}\right)$$

c) $$s(4) - s(1) = 6t^2 + 3t\Big|_1^4 = (96 + 12) - (6 + 3)$$
$$= 99$$

OR $$\int_{s(1)}^{s(4)} |\beta'(s)|\,ds = \int_9^{108} 1\cdot ds = 99$$

#2 $\alpha(t) = \left(-2t, t^2, \frac{1}{3}t^3\right)$

$$\alpha'(t) = (-2, 2t, t^2)$$
$$\alpha''(t) = (0, 2, 2t)$$
$$\alpha'''(t) = (0, 0, 2)$$

$$|\alpha'(t)| = \sqrt{4 + 4t^2 + t^4} = 2 + t^2 = v$$

$$T = \frac{\alpha'(t)}{v} = \frac{1}{2+t^2}(-2, 2t, t^2)$$

$$\alpha' \times \alpha'' = \begin{vmatrix} i & j & k \\ -2 & 2t & t^2 \\ 0 & 2 & 2t \end{vmatrix}$$

$$= (2t^2, 4t, -4)$$

$$|\alpha' \times \alpha''| = \sqrt{4t^4 + 16t^2 + 16} = 2\sqrt{t^4 + 4t^2 + 4}$$

$$= 2(t^2 + 2)$$

$$\kappa = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} = \frac{2(2+t^2)}{(2+t^2)^3} = \frac{2}{(2+t^2)^2}$$

$$B = \frac{\alpha' \times \alpha''}{|\alpha' \times \alpha''|} = \frac{1}{2(t^2+2)}(2t^2, 4t, -4)$$

$$= \frac{1}{t^2+2}(t^2, 2t, -2)$$

#2

②

$$N = B \times T = \begin{vmatrix} i & j & k \\ t^2 & 2t & -2 \\ -2 & 2t & t^2 \end{vmatrix} \cdot \frac{1}{t^2+2} \frac{1}{t^2+2}$$

$$= \left(2t^3 + 4t,\ 4 - t^4,\ 2t^3 + 4t \right) \cdot \frac{1}{(t^2+2)^2}$$

$$= \left(2t,\ 2 - t^2,\ 2t \right) \cdot \frac{1}{t^2+2}$$

$$1 \stackrel{?}{=} |N| = \frac{1}{t^2+2} \sqrt{4t^2 + 4 - \cancel{4t^2} + t^4 + \cancel{2t^4}} = 1 \checkmark$$

$$\tau = \frac{\alpha''' \cdot (\alpha' \times \alpha'')}{|\alpha' \times \alpha''|^2}$$

$$= \frac{(0,0,2) \cdot (2t^2, 4t, -4)}{4(t^2+2)^2} = \frac{-8}{4(t^2+2)^2}$$

$$\tau = \frac{-2}{(t^2+2)^2}$$

#2

Method II

$$\alpha(t) = \left(-2t,\ t^2,\ \tfrac{1}{3}t^3\right)$$

$$\alpha'(t) = \left(-2,\ 2t,\ t^2\right)$$

$$|\alpha'| = \sqrt{4 + 4t^2 + t^4} = 2 + t^2 = v.$$

$$T = \frac{\alpha'}{|\alpha'|} = \frac{1}{2+t^2}\left(-2, 2t, t^2\right)$$

$$v\kappa N = T' = \frac{-2t}{(2+t^2)^2}\left(-2, 2t, t^2\right) + \frac{1}{(2+t^2)}\left(0, 2, 2t\right)$$

$$= \frac{1}{(2+t^2)^2}\left[-2t\left(-2, 2t, t^2\right) + (2+t^2)\left(0, 2, 2t\right)\right]$$

$$= \frac{1}{(2+t^2)^2}\left[\left(4t,\ -4t^2 + 4 + 2t^2,\ -2t^3 + 2t^3 + 4t\right)\right]$$

$$= \frac{1}{(2+t^2)^2}\left[\left(4t,\ 4 - 2t^2,\ 4t\right)\right]$$

$$\kappa N = \frac{1}{(2+t^2)^3}\left[\left(4t,\ 4-2t^2,\ 4t\right)\right]$$

$$\kappa = |\kappa N| = \frac{1}{(2+t^2)^3}\cdot\left[\cancel{16t^2} + 16 - \cancel{16t^2} + 4t^4 + 16t^2\right]^{1/2}$$

$$= \frac{1}{(2+t^2)^3}\left[4\left(t^2+2\right)^2\right]^{1/2}$$

$$= \frac{2}{(2+t^2)^2}$$

$$N = \frac{(2+t^2)^2}{2}\cdot\frac{1}{(2+t^2)^3}\left[\left(4t,\ 4-2t^2,\ 4t\right)\right] = \frac{1}{2+t^2}\left(2t,\ 2-t^2,\ 2t\right)$$

#2

$$B = T \times N$$

$$= \begin{vmatrix} i & j & k \\ -2 & 2t & t^2 \\ 2t & 2-t^2 & 2t \end{vmatrix} \frac{1}{2+t^2} \frac{1}{2+t^2}$$

$$= \frac{1}{(2+t^2)^2} \left(4t^2 - 2t^2 + t^4,\ 2t^3 + 4t,\ -4 + 2t^2 - 4t^2 \right)$$

$$= \frac{1}{(2+t^2)^2} \left(t^4 + 2t^2,\ 2t(t^2+2),\ -4 - 2t^2 \right)$$

$$= \frac{1}{2+t^2} \left(t^2,\ 2t,\ -2 \right)$$

$$-\tau v = B' \cdot N = \left(\frac{-2t}{(2+t^2)^2} (t^2, 2t, -2) + \frac{1}{2+t^2} (2t, 2, 0) \right)$$

$$\cdot \frac{1}{2+t^2} (2t, 2-t^2, 2t)$$

$$= \frac{-2t}{(2+t^2)^3} \cdot \left(\cancel{2t^3} + \cancel{4t} - \cancel{2t^3} - \cancel{4t} \right) +$$

$$\frac{1}{(2+t^2)^2} \left(4t^2 + 4 - 2t^2 + 0 \right)$$

$$= \qquad = \frac{1}{(2+t^2)^2} \left(2t^2 + 4 \right)$$

$$-\tau \cdot \underbrace{(2+t^2)}_{v} = \frac{2}{2+t^2}$$

$$\tau = \frac{-2}{(2+t^2)^2}$$

#3

We are looking for a vector w satisfying

$T' = w \times T$
$N' = w \times N$
$B' = w \times B$ (if such w exists).

Since for any given $s = s_0$, $\{T(s_0), N(s_0), B(s_0)\}$ is an orthonormal frame, we can write

$w(s_0) = aT(s_0) + bN(s_0) + cB(s_0)$

(The choices of a, b and c will depend on s_0, and we will omit s_0 for now.)

$$w = aT + bN + cB$$

$T' = \kappa N = w \times T = (aT + bN + cB) \times T = -bB + cN$
$N' = -\kappa T + \tau B = w \times N = (aT + bN + cB) \times N = aB - cT$
$B' = -\tau N = w \times B = (aT + bN + cB) \times B = -aN + bT$

In order to have $w(s_0)$ exists, we must have

$$\left.\begin{array}{ll} ① & \kappa N = -bB + cN \\ ② & -\kappa T + \tau B = aB - cT \\ ③ & -\tau N = -aN + bT \end{array}\right\} \text{ all satisfied.}$$

Existence: Taking $a = \tau(s_0)$, $b = 0$, $c = \kappa(s_0)$, we see that all three equations are satisfied.

uniqueness:
$$\left.\begin{array}{l} ①: 0 = (c-\kappa)N + (-b)B \\ ②: 0 = (-c+\kappa)T + (a-\tau)B \\ ③: 0 = bT + (\tau - a)N \end{array}\right\}$$ since T, N, B is linearly independent
$$\left.\begin{array}{l} c - \kappa = 0 \\ -b = 0 \\ a - \tau = 0 \end{array}\right\}$$ are the only solns.

#3 continue.

$$\omega(s_0) = \tau(s_0)T(s_0) + \kappa(s_0)B(s_0). \quad \forall s_0 \text{ fixed}$$

Hence $\omega = \tau T + \kappa B$ exists for all s.

$$T' = \kappa N$$

$$T'' = \kappa' N + \kappa N' = \kappa' N + \kappa(-\kappa T + \tau B)$$

$$= -\kappa^2 T + \kappa' N + \kappa\tau B.$$

$$T' \times T'' = (\kappa N) \times (-\kappa^2 T + \kappa' N + \kappa\tau B)$$

$$= -\kappa^3 \underbrace{(N \times T)}_{-B} + \kappa\kappa' \underbrace{(N \times N)}_{0} + \kappa^2\tau \underbrace{(N \times B)}_{T}$$

$$= \kappa^2\tau T + \kappa^3 B$$

$$= \kappa^2 \underbrace{(\tau T + \kappa B)}_{\omega} = \kappa^2 \omega$$

#4

Exc 1.3.24

$\left(\frac{1}{k}\right)' \neq 0 \qquad \underbrace{\left(\frac{1}{k}\right)^2 + \left(\left(\frac{1}{k}\right)'\frac{1}{\tau}\right)^2 = \text{constant} = R^2}_{*}$

$\Rightarrow \quad \alpha \subseteq S_r(p_0)$

Sol'n

Let $f(s) = \alpha(s) + \frac{1}{k}N + \left(\frac{1}{k}\right)'\frac{1}{\tau}B.$

$$f'(s) = T + \left(\frac{1}{k}\right)'N + \frac{1}{k}(-kT + \tau B) + \left(\left(\frac{1}{k}\right)'\left(\frac{1}{\tau}\right)\right)'B + \left(\frac{1}{k}\right)'\frac{1}{\tau}(-\tau N)$$

$$= \cancel{T} + \cancel{\left(\frac{1}{k}\right)'N} - \cancel{T} + \frac{\tau}{k}B + \left(\left(\frac{1}{k}\right)'\left(\frac{1}{\tau}\right)\right)'B - \cancel{\left(\frac{1}{k}\right)'N}$$

$$= B\Big(\underbrace{\frac{\tau}{k} + \left(\left(\frac{1}{k}\right)'\left(\frac{1}{\tau}\right)\right)'}_{=0}\Big) \Rightarrow f(s) = p_0 \text{ constant}$$

(1) since $\left(\frac{1}{k}\right)^2 + \left(\left(\frac{1}{k}\right)'\frac{1}{\tau}\right)^2 = \text{const} = R^2$

$$\Rightarrow 2\left(\frac{1}{k}\right)\left(\frac{1}{k}\right)' + 2\left(\left(\frac{1}{k}\right)'\left(\frac{1}{\tau}\right)\right)\cdot\left(\left(\frac{1}{k}\right)'\frac{1}{\tau}\right)' = 0$$

$$\left(\left(\frac{1}{k}\right)'\left(\frac{1}{\tau}\right)\right)' = -\frac{2\left(\frac{1}{k}\right)\left(\frac{1}{k}\right)'}{2\left(\frac{1}{\tau}\right)\left(\frac{1}{k}\right)'} = -\frac{\tau}{k}.$$

$$\|f(s) - \alpha(s)\|^2 = \left\|\frac{1}{k}N + \left(\frac{1}{k}\right)'\frac{1}{\tau}B\right\|^2 = \left(\frac{1}{k}\right)^2 + \left(\left(\frac{1}{k}\right)'\frac{1}{\tau}\right)^2 = R^2 \text{ constant}.$$

#4 Exc 1.3.24

Another solution.

Let $\alpha(s)$ be given, $\tau \neq 0$, $\left(\frac{1}{k}\right)' \neq 0$.

Take $p(s) = \alpha(s) + \frac{1}{k} N + \left(\frac{1}{k}\right)' \frac{1}{\tau} B$

$$\| p(s) - \alpha(s) \|^2 = \left\| \frac{1}{k} N + \left(\frac{1}{k}\right)' \frac{1}{\tau} B \right\|^2 = \underbrace{\left(\frac{1}{k}\right)^2 + \left(\left(\frac{1}{k}\right)' \frac{1}{\tau}\right)^2}_{\text{Given}} = \text{constant}$$

$$0 = \frac{d}{ds}\left((p(s) - \alpha(s)) \cdot (p(s) - \alpha(s)) \right)$$

$$2\left(p'(s) - \alpha'(s) \right) \cdot \left(p(s) - \alpha(s) \right) = 0.$$

$$2\left(p'(s) - T \right) \cdot \left(p(s) - \alpha(s) \right) = 0$$

$$T(s) \cdot \left(p(s) - \alpha(s) \right) = T \cdot \left(\frac{1}{k} N + \left(\frac{1}{k}\right)' \frac{1}{\tau} B \right)(s) = 0$$

$$\Longrightarrow \quad p'(s) \cdot \left(p(s) - \alpha(s) \right) = 0.$$

$$p' = T + \left(\frac{1}{k}\right)' N + \left(\frac{1}{k}\right)(-kT + \tau B) + \left(\left(\frac{1}{k}\right)' \frac{1}{\tau}\right)' B + \left(\frac{1}{k}\right)' \frac{1}{\tau} (-\tau N)$$

$$= \cancel{T} + \cancel{\left(\frac{1}{k}\right)' N} - \cancel{T} + \frac{\tau}{k} B + \left(\left(\frac{1}{k}\right)' \frac{1}{\tau}\right)' B - \cancel{\left(\frac{1}{k}\right)' N}$$

$$p' = \left(\frac{\tau}{k} + \left(\left(\frac{1}{k}\right)' \frac{1}{\tau}\right)' \right) \cdot B$$

$$p - \alpha = \frac{1}{k} N + \left(\frac{1}{k}\right)' \frac{1}{\tau} B$$

$$0 = p' \cdot (p - \alpha) = \underbrace{\left(\frac{\tau}{k} + \left(\left(\frac{1}{k}\right)' \left(\frac{1}{\tau}\right)\right)' \right)}_{\text{requires } 0} \underbrace{\left(\frac{1}{k}\right)'}_{\neq 0} \underbrace{\frac{1}{\tau}}_{\neq 0} \quad \Longrightarrow \quad p' = 0 \cdot B = 0$$

$$\Longrightarrow \quad p(s) \text{ constant.}$$

#5 Exc. 1.4.14

$\alpha(t) = (a\cos t, b\sin t, 0)$ We will do the calculations in $\mathbb{R}^3$.

$\alpha' = (-a\sin t, b\cos t, 0) \rightarrow |\alpha'| = v = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$

$\alpha'' = (-a\cos t, -b\sin t, 0)$ $\quad T = \frac{1}{v}\alpha'$

$\alpha' \times \alpha'' = (0, 0, ab) \implies \kappa = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} = \frac{ab}{v^3}$

$$B = \frac{\alpha' \times \alpha''}{|\alpha' \times \alpha''|} = (0,0,1)$$

$$N = B \times T = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ -a\sin t & b\cos t & 0 \end{vmatrix} \frac{1}{v} = \frac{1}{v}(-b\cos t, -a\sin t, 0)$$

Now go back to $\mathbb{R}^2$:

$$\mathcal{E} = \alpha + \frac{1}{\kappa}N = (a\cos t, b\sin t) + \frac{v^3}{ab}\frac{1}{v}(-b\cos t, -a\sin t)$$

$$= \left((\cos t)\left(a - \frac{v^2}{a}\right), (\sin t)\left(b - \frac{v^2}{b}\right)\right)$$

$$a - \frac{v^2}{a} = \frac{1}{a}\left(\underbrace{a^2 - a^2\sin^2 t}_{a^2\cos^2 t} - b^2\cos^2 t\right) = \frac{a^2 - b^2}{a}\cos^2 t$$

$$b - \frac{v^2}{b} = \frac{1}{b}\left(\underbrace{b^2 - b^2\cos^2 t}_{= b^2\sin^2 t} - a^2\sin^2 t\right) = \frac{b^2 - a^2}{b}\sin^2 t$$

$$\mathcal{E}(t) = \left(\frac{a^2 - b^2}{a}\cos^3 t, \frac{b^2 - a^2}{b}\sin^3 t\right)$$

#6

$$|\alpha'(s)| = 1, \quad \kappa_\alpha > 0, \quad \tau_\alpha > 0$$

$$\beta = T_\alpha = \alpha'$$

$$\underbrace{v_\beta}_{>0} \underbrace{T_\beta}_{\text{unit}} = \beta' = T_\alpha' = \alpha'' = \underbrace{\kappa_\alpha}_{>0} \underbrace{N_\alpha}_{\text{unit}} \implies \begin{array}{l} v_\beta = \kappa_\alpha = |\beta'| \\ T_\beta = N_\alpha . \end{array}$$

$$\begin{aligned} \beta'' &= \kappa_\alpha' N_\alpha + \kappa_\alpha N_\alpha' = \\ &= \kappa_\alpha' N_\alpha + \kappa_\alpha(-\kappa_\alpha T_\alpha + \tau_\alpha B_\alpha) \\ &= -\kappa_\alpha^2 T_\alpha + \kappa_\alpha' N_\alpha + \kappa_\alpha \tau_\alpha B_\alpha . \end{aligned}$$

$$\begin{aligned} \beta' \times \beta'' &= \kappa_\alpha N_\alpha \times (-\kappa_\alpha^2 T_\alpha + \kappa_\alpha' N_\alpha + \kappa_\alpha \tau_\alpha B_\alpha) \\ &= -\kappa_\alpha^3(-B_\alpha) + 0 + \kappa_\alpha^2 \tau_\alpha T_\alpha . \\ &= \kappa_\alpha^2 \tau_\alpha T_\alpha + \kappa_\alpha^3 B_\alpha \\ &= \kappa_\alpha^2 (\tau_\alpha T_\alpha + \kappa_\alpha B_\alpha) \end{aligned}$$

$$\kappa_\beta = \frac{|\beta' \times \beta''|}{|\beta'|^3} = \frac{\kappa_\alpha^2 \sqrt{\tau_\alpha^2 + \kappa_\alpha^2}}{\kappa_\alpha^3} = \frac{1}{\kappa_\alpha} \sqrt{\tau_\alpha^2 + \kappa_\alpha^2} = \sqrt{1 + \left(\frac{\tau_\alpha}{\kappa_\alpha}\right)^2} .$$

$$B_\beta = \frac{\beta' \times \beta''}{|\beta' \times \beta''|} = \frac{1}{\sqrt{\tau_\alpha^2 + \kappa_\alpha^2}} (\tau_\alpha T_\alpha + \kappa_\alpha B_\alpha) = B_\beta$$

$$N_\beta = B_\beta \times T_\beta = \begin{vmatrix} T_\alpha & N_\alpha & B_\alpha \\ \tau_\alpha & 0 & \kappa_\alpha \\ 0 & 1 & 0 \end{vmatrix} \frac{1}{\sqrt{\tau_\alpha^2 + \kappa_\alpha^2}} = \frac{1}{\sqrt{\tau_\alpha^2 + \kappa_\alpha^2}} (-\kappa_\alpha T_\alpha + \tau_\alpha B_\alpha)$$

#6

$$\beta'' = -\kappa_\alpha^2 T_\alpha + \kappa_\alpha' N_\alpha + \kappa_\alpha \tau_\alpha B_\alpha$$

$$\beta''' = -2\kappa_\alpha \kappa_\alpha' T_\alpha - \kappa_\alpha^2 (\kappa_\alpha N_\alpha) + \\ + \kappa_\alpha'' N_\alpha + \kappa_\alpha' (-\kappa_\alpha T_\alpha + \tau_\alpha B_\alpha) \\ + (\kappa_\alpha' \tau_\alpha + \kappa_\alpha \tau_\alpha') B_\alpha + \kappa_\alpha \tau_\alpha (-\tau_\alpha N_\alpha)$$

$$\beta''' = T_\alpha (-2\kappa_\alpha \kappa_\alpha' - \kappa_\alpha \kappa_\alpha') + \\ N_\alpha (-\kappa_\alpha^3 + \kappa_\alpha'' - \kappa_\alpha \tau_\alpha^2) + \\ B_\alpha (\kappa_\alpha' \tau_\alpha + \kappa_\alpha' \tau_\alpha + \kappa_\alpha \tau_\alpha')$$

$$\beta' \times \beta'' = \kappa_\alpha^2 \tau_\alpha T_\alpha + 0 \cdot N_\alpha + \kappa_\alpha^3 B_\alpha .$$

$$(\beta' \times \beta'') \cdot \beta''' = -3\kappa_\alpha \kappa_\alpha' \cdot \kappa_\alpha^2 \tau_\alpha + 0 + 2\kappa_\alpha^3 \kappa_\alpha' \tau_\alpha + \kappa_\alpha^4 \tau_\alpha'$$

$$= -3\kappa_\alpha^3 \kappa_\alpha' \tau_\alpha + 2\kappa_\alpha^3 \kappa_\alpha' \tau_\alpha + \kappa_\alpha^4 \tau_\alpha'$$

$$= -\kappa_\alpha^3 \kappa_\alpha' \tau_\alpha + \kappa_\alpha^4 \tau_\alpha'$$

$$= \kappa_\alpha^3 (\kappa_\alpha \tau_\alpha' - \kappa_\alpha' \tau_\alpha)$$

$$\tau_\beta = \frac{\beta' \times \beta'' \cdot \beta'''}{|\beta' \times \beta''|^2} = \frac{\kappa_\alpha^3 (\kappa_\alpha \tau_\alpha' - \kappa_\alpha' \tau_\alpha)}{\kappa_\alpha^4 (\tau_\alpha^2 + \kappa_\alpha^2)} = \left(\frac{\kappa_\alpha \tau_\alpha' - \kappa_\alpha' \tau_\alpha}{\kappa_\alpha^2}\right)\left(\frac{\kappa_\alpha}{\tau_\alpha^2 + \kappa_\alpha^2}\right)$$

$$= \frac{\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'}{\kappa_\alpha \left(1 + \left(\frac{\tau_\alpha}{\kappa_\alpha}\right)^2\right)}$$

#6 Another way to calculate τ_β.

$$B_\beta = \frac{1}{\sqrt{\tau_\alpha^2+\kappa_\alpha^2}}\left(\tau_\alpha T_\alpha + \kappa_\alpha B_\alpha\right) = \frac{1}{\sqrt{1+\sigma^2}}\left(\sigma T_\alpha + B_\alpha\right)$$

$$\boxed{\text{Let } \frac{\tau_\alpha}{\kappa_\alpha} = \sigma}$$

$$N_\beta = \frac{1}{\sqrt{\tau_\alpha^2+\kappa_\alpha^2}}\left(-\kappa_\alpha T_\alpha + \tau_\alpha B_\alpha\right) = \frac{1}{\sqrt{1+\sigma^2}}\left(-T_\alpha + \sigma B_\alpha\right)$$

$$B_\beta' = -\tau_\beta v_\beta N_\beta$$

$$B_\beta' = -\frac{1}{2}(1+\sigma^2)^{-\frac{3}{2}}\cdot 2\sigma\sigma'\left(\sigma T_\alpha + B_\alpha\right)$$
$$+\ (1+\sigma^2)^{-\frac{1}{2}}\cdot\left(\sigma' T_\alpha + \sigma\kappa_\alpha N_\alpha - \tau_\alpha N_\alpha\right)$$
$$= (1+\sigma^2)^{-\frac{3}{2}}\Big[-\sigma^2\sigma' T_\alpha - \sigma\sigma' B_\alpha$$
$$+ (1+\sigma^2)(\sigma' T_\alpha + \sigma\kappa_\alpha N_\alpha - \tau_\alpha N_\alpha)\Big]$$

$$= (1+\sigma^2)^{-\frac{3}{2}}\Big[T_\alpha\left(-\cancel{\sigma^2\sigma'} + \sigma' + \cancel{\sigma^2\sigma'}\right) + N_\alpha\big(\underbrace{\sigma\kappa_\alpha + \sigma^3\kappa_\alpha - \tau_\alpha - \sigma^2\tau_\alpha}_{0}\big) + B_\alpha\left(-\sigma\sigma'\right)\Big]$$

$$\sigma\kappa_\alpha + \sigma^3\kappa_\alpha - \tau_\alpha - \sigma^2\tau_\alpha = \frac{\tau}{\kappa}\cdot\cancel{\kappa} + \frac{\tau^3}{\cancel{\kappa^3}}\cancel{\kappa} - \cancel{\tau} - \frac{\tau^2}{\cancel{\kappa^2}}\cancel{\tau} = 0$$

$$B_\beta' = -\tau_\beta v_\beta N_\beta = (1+\sigma^2)^{-\frac{3}{2}}\cdot\sigma'\left[T_\alpha - \sigma B_\alpha\right]$$

$$= \frac{\sigma'}{1+\sigma^2}\cdot\frac{T_\alpha - \sigma B_\alpha}{(1+\sigma^2)^{1/2}} = \frac{\sigma'}{1+\sigma^2}(-N_\beta)$$

$$\tau_\beta = +\frac{1}{v_\beta}\frac{\sigma'}{1+\sigma^2} = +\frac{1}{\kappa_\alpha}\frac{\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'}{\left(1+\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)^2\right)}$$

#7

a) $\beta = \alpha - r N_\alpha$ $\qquad$ $\tau = 0$ plane curve

$\beta' = T_\alpha - r(-\kappa_\alpha T_\alpha + \cancel{\tau_\alpha B_\alpha})$

$v_\beta T_\beta = \beta' = T_\alpha(1 + r\kappa_\alpha)$

$v_\beta = |1 + r\kappa_\alpha|$ $\qquad$ $T_\beta = T_\alpha$ when $1 + r\kappa_\alpha > 0$

$T_\beta = -T_\alpha$ when $1 + r\kappa_\alpha < 0$

β not regular when $1 + r\kappa_\alpha = 0$

$\beta'' = (\kappa_\alpha N_\alpha)(1 + r\kappa_\alpha) + T_\alpha(+\kappa_\alpha' r)$

$= +\kappa_\alpha' r\, T_\alpha + \kappa_\alpha(1 + r\kappa_\alpha) N_\alpha$

$\beta' = (1 + r\kappa_\alpha) T_\alpha$

$\beta' \times \beta'' = \kappa_\alpha (1 + r\kappa_\alpha)^2 \underbrace{T_\alpha \times N_\alpha}_{B_\alpha}$

$|\beta' \times \beta''| = \kappa_\alpha (1 + r\kappa_\alpha)^2$

$$\kappa_\beta = \frac{|\beta' \times \beta''|}{|\beta'|^3} = \frac{\kappa_\alpha (1 + r\kappa_\alpha)^2}{|1 + r\kappa_\alpha|^3} = \frac{\kappa_\alpha}{|1 + r\kappa_\alpha|}$$

(b) $\alpha(s) = (\cos s, \sin s)$

$T = \alpha' = (-\sin s, \cos s)$

$N = \kappa N = \alpha'' = (-\cos s, -\sin s)$

Take $r = -1$: $\alpha + 1 \cdot N_\alpha = (0, 0) = \beta(s)$ $\quad$ pt curve, not regular

In fact, for any $\alpha(s)$, if $\kappa(s_0) \neq 0$, $\beta(s) = \alpha(s) + \frac{1}{\kappa(s_0)} N(s)$ is not regular at $s = s_0$, since $v_\beta(s_0) = 0$

(c)

$$\text{length of } \beta = \int_0^L |\beta'(s)|\,ds$$
$$= \int_0^L |1 + r k_\alpha|\,ds$$

If $r \geq 0$, then
$1 + r k_\alpha > 0$
$|1 + r k_\alpha| = 1 + r k_\alpha$.

$$0 > r > -(\max k_\alpha)^{-1} = -\frac{1}{\max k_\alpha}.$$
$$(\max k_\alpha) r > -1.$$
$$1 + (\max k_\alpha) r > 0$$
$$k_\alpha \leq \max k_\alpha$$
$$r k_\alpha \geq r \max k_\alpha.$$
$$1 + r k_\alpha \geq 1 + r \max k_\alpha > 0.$$
$$|1 + r k_\alpha| = 1 + r k_\alpha.$$

$$L(\beta) = \int_0^L 1 + r k_\alpha$$
$$= L + r \int_0^L k_\alpha\,ds. = L(\alpha) + r \cdot \int_0^L \tilde{k}_\alpha\,ds$$

$\int_0^L |\alpha'(s)|\,ds = L = L(\alpha)$

$$= L(\alpha) + r \cdot 2\pi \underbrace{I(\alpha)}_{1}$$
$$= L(\alpha) + 2\pi r.$$

Why $k_\alpha = \tilde{k}_\alpha$

since $k_\alpha > 0 \implies |\tilde{k}_\alpha| = k_\alpha \neq 0$

$\implies \tilde{k}_\alpha > 0\ \forall s$ OR $\tilde{k}_\alpha < 0\ \forall s$.

$\tilde{k}_\alpha < 0 \implies \alpha$ rotates clockwise.

$\implies \tilde{k}_\alpha > 0\ \forall s \implies k_\alpha = \tilde{k}_\alpha\ \forall s$.