MATH 4500
Take-home Final
Due December 13, 2017 by 12 noon in my office B20F MLH.

You are allowed to use any theorem, proof, calculation or example done in class or in the sections of the textbook we covered so far, as long as you clearly indicate what you are using. If you want to use a computer to check some graphs of curves or calculations, it is fine. You still need to show details of your calculations and proofs written by you. For a take-home test, students are allowed to use their notes, textbook, to ask your instructor questions/hints in class or during office hours. Discussing the test questions with other students is **not** allowed. Using another person's solutions is **not** allowed. Using other textbooks or online resources is **not** allowed.

If a question has hints, then you may choose to use them or not. However, a hint is not a part of the hypothesis, and you still are required to substantiate or prove the hints as any other statement you claim in your solutions.

Since this is take home-exam, your solutions should be written legibly, showing details and steps, and your answers should be simplified as much as possible. Please include a cover page and start with a new page with each question. Doing these will give me space to write grades and comments. You can write on both sides of the paper if you would like. Either staple your pages together or write your name on each page.

There are 7 questions in this test.
First question is 10 points and the rest are 15 points each, for a total of 100 points. Questions 1-5 are exercises from the textbook of the course, by Oprea. Some of the questions are slightly changed. Do them as they are stated in this exam.

1. Exercise 2.1.25, p. 76.

Find the lines of striction for the helicoid, and the hyperboloid of one sheet. For the latter, assume that for convenience that $x^2 + y^2 - z^2 = 1$.

2. Exercise 2.2.15, p. 85. Caution: The signs in the textbook are supposed to be opposite.

Show that the shape operator S for the torus $\mathbf{x}(u,v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u)$ with respect to the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ is given by

$$S(\mathbf{x}_u) = \frac{1}{r}\mathbf{x}_u \text{ and } S(\mathbf{x}_v) = \frac{\cos u}{R + r\cos u}\mathbf{x}_v$$

3. Exercise 3.3.7a, p. 121.

For the surface of revolution obtained by revolving $\alpha(u) = (u, e^{-\frac{1}{2}u^2}, 0)$ about the x-axis, find K and describe where $K > 0, K < 0, K = 0$.

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4. Exercise 3.2.15, p. 116, with the first and second fundamental forms.

Find the Gauss and mean curvatures of the helicoid given by $\mathbf{x}(u,v) = (u\cos v, u\sin v, bv)$, where $b \in \mathbb{R}$, and provide the components of the first and second fundamental forms: E, F, G, l, m and n.

5. The total torsion of a unit speed curve $\alpha(s) : [a,b] \to \mathbb{R}^3$ is $\int_a^b \tau ds$ provided that $\kappa_\alpha > 0$. Show that a smoothly closed unit speed curve α on a sphere S^2 of radius 1 has zero total torsion, provided that $\kappa_\alpha > 0$ and $\alpha''(s)$ is never normal to S^2 at $\alpha(s)$. It is not clear in the textbook, but one must assume α is smoothly closed.

HINT. This is a weaker version of Exercise 2.4.2, p.93, by assuming $\theta \neq 0, \pi$ (that is: α'' is never normal to S^2) on $[a,b]$ and taking $R = 1$. In addition to the hints in the book, also show that one can take $U(\alpha(s)) \ = \alpha(s) = N_\alpha \cos\theta + B_\alpha \sin\theta$ where N_α and B_α are the principal normal and the binormal of α in $\mathbb{R}^3$.

6. Let M be a regular surface in $\mathbb{R}^3$ with $H \equiv 0$. Prove that for all $p \in M$ and for all $v, w \in T_pM$

$$S_p(v) \cdot S_p(w) = -K(p)(v \cdot w).$$

7.

a. Calculate the geodesic curvature of $\gamma_m(t) = (t, mt, mt^2)$ in the surface $M = \{(x,y,z) \in \mathbb{R}^3 : z = xy\}$, for each fixed $m \in \mathbb{R}$.

b. **i**. Find all $m \in \mathbb{R}$ for which the geodesic curvature of $\gamma_m(t)$ in M is 0 for all $t \in \mathbb{R}$.

ii. Which of these $\gamma_m(t)$ are geodesics of M for all $t \in \mathbb{R}$?

iii. Find all normal sections of M at $(0,0,0)$, which can be parametrized as a geodesic of M for all $t \in \mathbb{R}$.