Dec 6, 2017

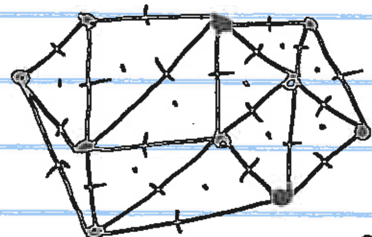
An Informal preview of Gauss-Bonnet Thm.

(1) Combinatorial Euler Characteristic

Examples

(1)

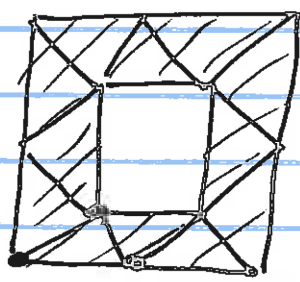
2-Disc



$V = 10$
$E = 20$
$F = 11$

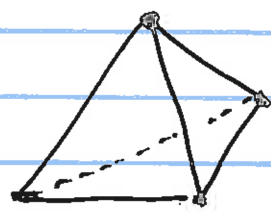
$$\chi(D) = V - E + F = 10 - 20 + 11 = 1.$$

(2)



$\chi = 0$ Annulus

(3)

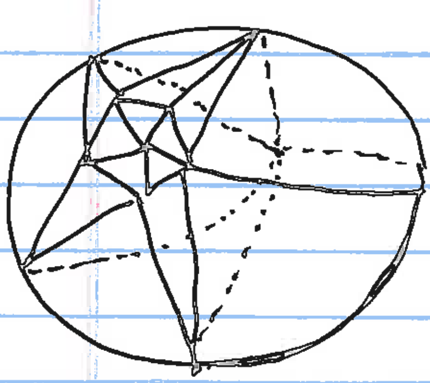


Tetrahedron

$V = 4$
$E = 6$
$F = 4$

$\chi = 2$

S^2

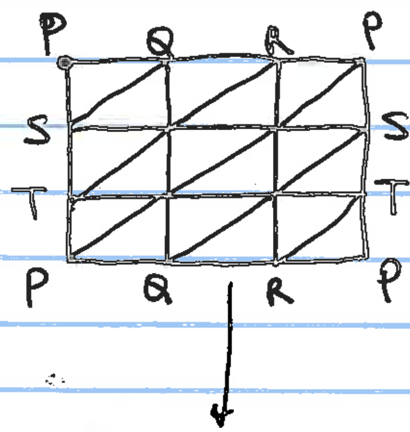


$\rightarrow \chi = 2$

for any proper triangulation.

χ is defined by using Homology, etc.
So it is a "topological invariant"
When M has a proper triangulation, $V-E+F$ will always be equal to topologically obtained $\chi(M)$ provided that M is a regular 2-surface.

Another Ex ① T^2

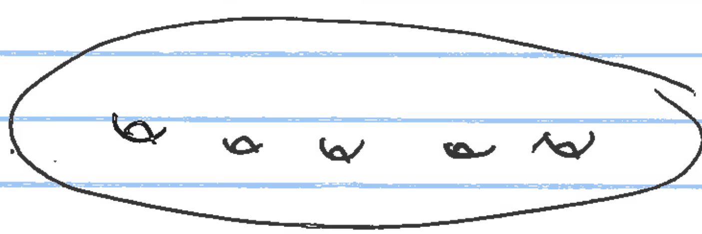


recall identifying left & right edges, identifying upper & lower edges.
All P's are identified
All S's " "
" T " " etc.

$V = 9$
$E = 33 - 6 = 27$
$F = 18$

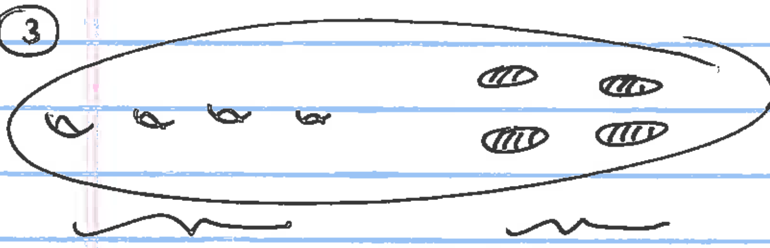
For Torus $\chi = 9 - 27 + 18 = 0$

②



$\chi = 2 - 2g$

③

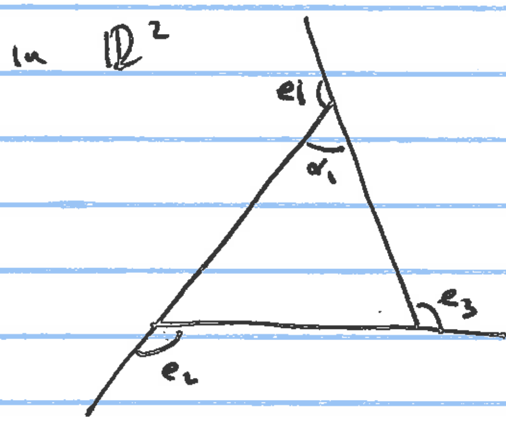


$\chi = 2 - 2g - h.$

genus g $h = \#$ punctured discs

Angle Excess:

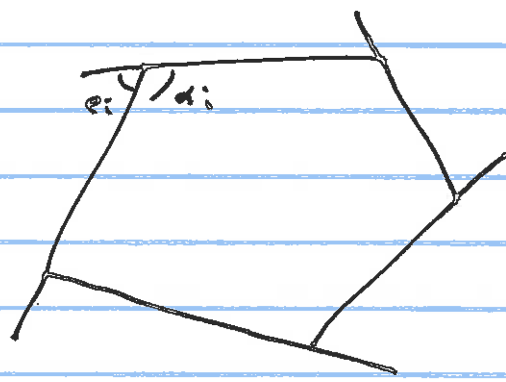
in $\mathbb{R}^2$



$$\alpha_1 + \alpha_2 + \alpha_3 = \pi$$

$$e_1 + e_2 + e_3 = 2\pi$$

α_i internal angles
e_i external angles

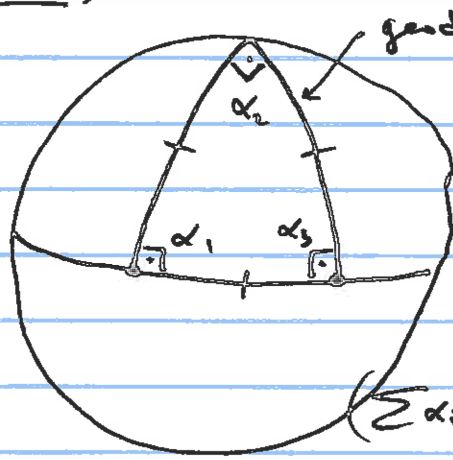


$$\sum e_i = 2\pi$$

$$\sum \alpha_i = \pi(n-2)$$

edges

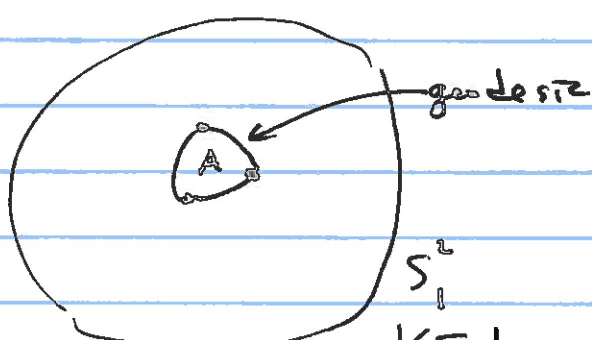
On S^2, with radius 1.



α_i internal ∡
e_i external ∡

$$\sum e_i = \frac{3\pi}{2} = \sum \alpha_i$$

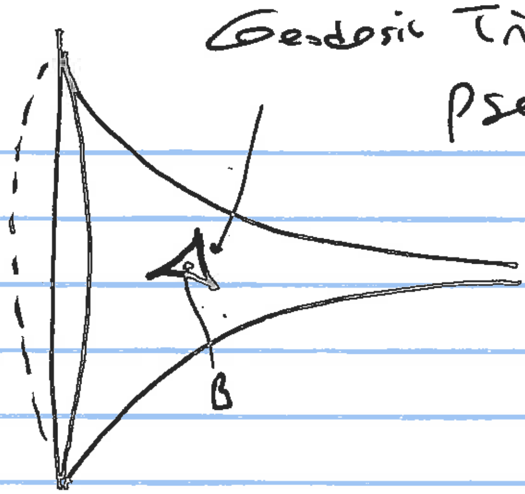
$$\left(\sum \alpha_i\right) - \pi = \frac{3\pi}{2} - \pi = \text{Excess} = \frac{\pi}{2}$$



$$\sum \alpha_i - \pi = \text{area } A = \text{exces}$$

S^2_1

$K \equiv 1.$



Caution

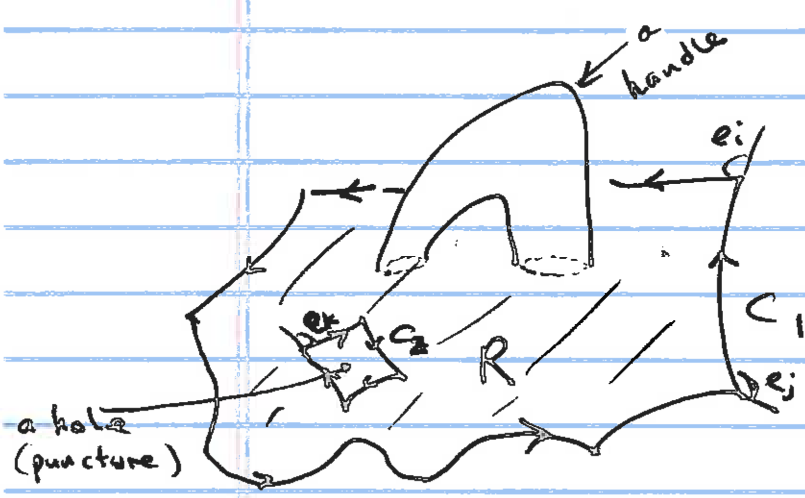
Typo: Need α_i.

$\sum \alpha_i < \pi$

$\sum \alpha_i - \pi = -$ area B.

THM (GAUSS-BONNET)

Let R be a regular region $\subseteq M^2$, a regular oriented surface. Let $C_1, C_2, \ldots C_m$ be closed simple, piecewise regular, disjoint curves whose union is ∂R, and positively oriented. Let $e_1, e_2, \ldots e_p$ be the external angles of $C_1, C_2, \ldots C_m$. Then

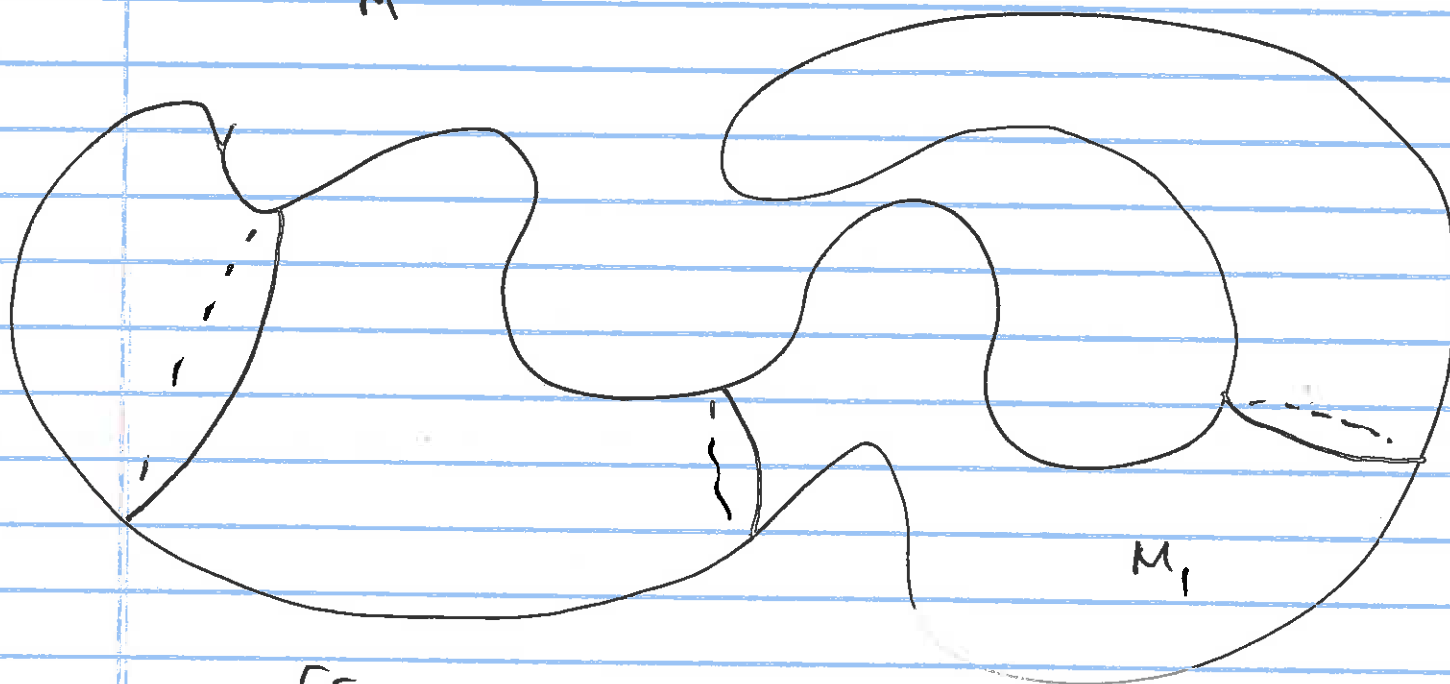


Then:

$$\sum_{i=1}^{m} \int_{C_i} k_g(s)\,ds + \sum_{i=1}^{p} e_i + \iint_R K\,dS = 2\pi\,\chi(R)$$

<u>Corollary</u> If M is compact, $\partial M = \emptyset$, oriented
(e.g. sphere, torus, M_g

$$\iint_M K\,dS = 2\pi\chi(M).$$



$$\iint_{M_1} K\,dS = 4\pi$$

M_1 is homeomorphic to S^2

M_2 is homeomorphic to T^2.

$$\iint_{M_2} K\,dS = 0$$

S^2 $\quad \chi(S^2) = 2 \quad \exists$ metric g s.t. $K \equiv 1$

S^2_1. $\quad \iint_{S^2} 1 \, dS = 4\pi$

area of S^2

T^2 $\quad \chi(T^2) = 0 \quad \exists$ metric g s.t. $K \equiv 0$

Flat torus $T^2 \subseteq \mathbb{R}^4$.

If $T^2 \subseteq \mathbb{R}^3$

Thm from 3.5, T^2 compact $\subseteq \mathbb{R}^3$

$\Rightarrow \exists R > 0 \quad T^2 \subseteq B_R(\vec{0})$

$\Rightarrow \exists p \in T^2 \quad K(p) \geq \frac{1}{R^2} > 0$.

$\iint K \, dS = 0$ $\quad$ if $\exists p$ $K(p) > 0$, then there must exist a point q s.t. $K(q) < 0$. (Otherwise, $\iint K \, dS > 0$, which is not the case.)

$\exists r \in T^2$ s.t. $K(r) = 0$, by Intermediate value Thm, since T^2 is connected.