Thm: (Clairaut Relation)

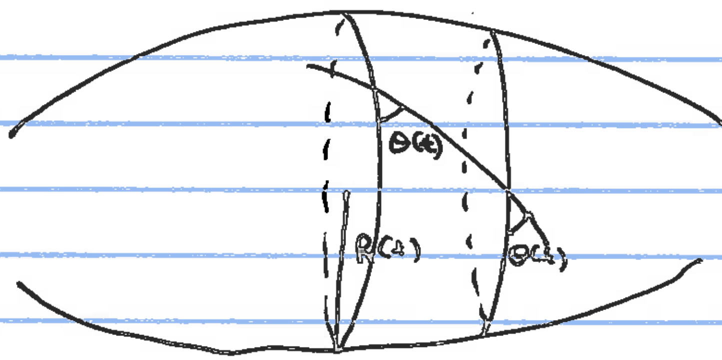
Let $\alpha(t)$ be a geodesic of a surface of revolution, $|\alpha'(t)| = 1$.

Let $\theta = \measuredangle(\alpha'(t), \varphi_\theta(\alpha(t)))$, i.e. the angle $\alpha'(t)$ makes with the parallel thru $\alpha(t)$.

Then

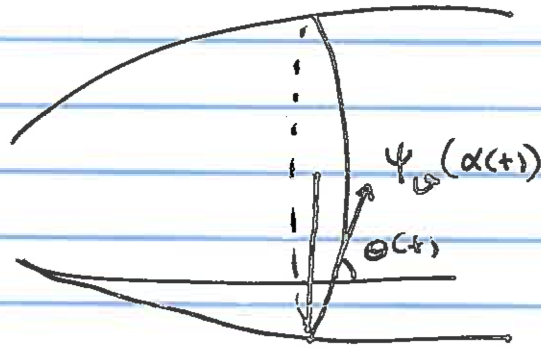
$$R(\alpha(t))\cos\theta(t) \equiv \text{constant} \quad \forall t.$$

where $R = h$ (distance to the rotation axis.)



Proof on Friday

Proof of Clairaut's relation



α geodesic, $|\alpha'(t)| = 1$

$R(t) = h(\alpha(t))$

$\alpha(t) = \psi(\underbrace{u(t), v(t)}_{\tilde{\alpha}})$

$$\alpha'(t) \cdot \psi_v(\alpha(t)) = \cos\theta(t) \cdot \underbrace{|\alpha'(t)|}_{1} \cdot \underbrace{|\psi_v(\alpha(t))|}_{h.}$$

$$\left.\begin{aligned} \psi_u \cdot \psi_u &= E = \sigma^2 \\ |\psi_v|^2 = \psi_v \cdot \psi_v &= G = h^2 \\ \psi_u \cdot \psi_v &= F = 0 \end{aligned}\right\} \Longrightarrow |\psi_v| = h$$

$$h(\alpha(t)) \cdot \cos\theta(t) = \alpha'(t) \cdot \psi_v(\alpha(t))$$

$$= (\underbrace{\psi_u \cdot u' + \psi_v \cdot v'}_{\alpha'}) \cdot \psi_v(\alpha(t))$$

$$= 0 + h^2 \cdot v'$$

$$\Longrightarrow \underbrace{h\cos\theta(t)}_{\text{WTS constant}} = h^2 v'$$

remember
$(h = h(u(t)) = R(t))$

$$\frac{d}{dt}(h\cos\theta(t)) = \frac{d}{dt}\left(h^2(u(t)) \cdot v'(t)\right)$$

$$\frac{d}{dt}\left(h^2(u(t))\cdot \vartheta'(t)\right)$$

$$= 2h(u(t))\cdot h'(u(t))\cdot u'(t)\cdot \vartheta'(t) +$$

$$+ h^2(u(t))\cdot \vartheta''(t))$$

$$= h^2\Big(\underbrace{\vartheta'' + 2\frac{h'}{h}u'\vartheta'}_{0 \text{ by geodesic eqn } (2)}\Big) = 0.$$

$$\frac{d}{dt}\left(h\cos\theta\right) = 0 = \frac{d}{dt}\left(R(t)\cdot\cos\theta(t)\right)$$

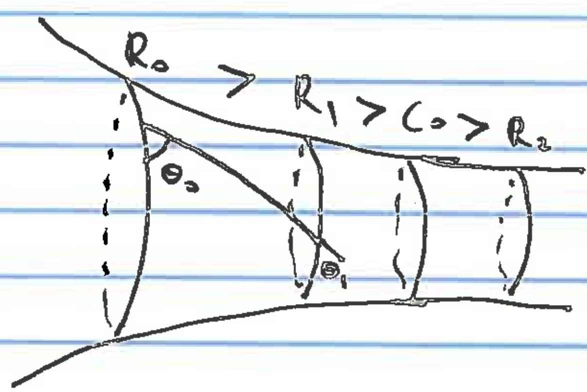
$R\cos\theta$ is Constant. #

$$R(t) = h(u(t)).$$

Obs 1 $\quad (\cos\theta(t))\cdot R(t) \equiv C_0.$

$$\Longrightarrow R(t) \geq C_0 \quad \left(|\cos\theta| \leq 1\right)$$

Obs 2



$$(\cos\theta_0)R_0 \equiv C_0$$
$$(\cos\theta_1)R_1 \equiv C_0$$

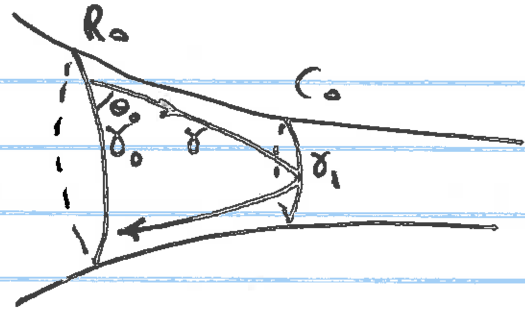
$$R_1 < R_0$$

$$\cos\theta_1 > \cos\theta_0$$

$$\theta_1 < \theta_0$$

Obs 3

Case 1



Assume C_0 is not at a neck

[C_0 is at u_0 with $h'(u_0) \neq 0$]

γ_1 is not a geodesic

$$\cos\theta_0 \cdot R_0 = \underbrace{C_0 \equiv \cos\theta \cdot C_0}$$

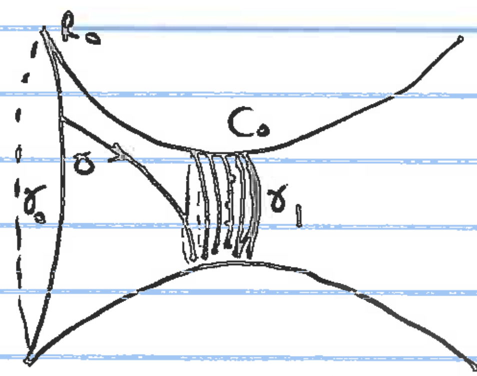
$$\Downarrow$$

$$\cos\theta = 1$$

$$\theta = 0$$

γ is tangential to γ_1, (which is not a geodesic); ($\uparrow$ geodesic) then γ turns around to come back to γ_0

Case 2 $\quad R_1 = C_0$ is a local min

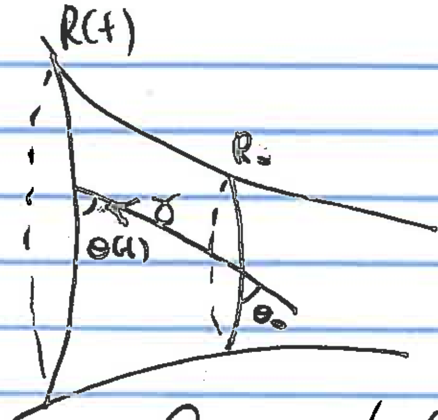


γ_1 is a geodesic

If γ reaches to γ_1 then it will be tangential to γ_1, but this contradicts Thm III (about the uniqueness of geodesics which are tangential). γ can't reach γ_1.

γ will converge to γ_1 in a spiral, but never reaches γ_1. The slope of γ', i.e. $\theta(t)$ satisfy $\theta(t) \searrow 0$, but $\theta(t) > 0 \ \forall t \geq t_0$.

Obs 4



$R(t) > R_0$

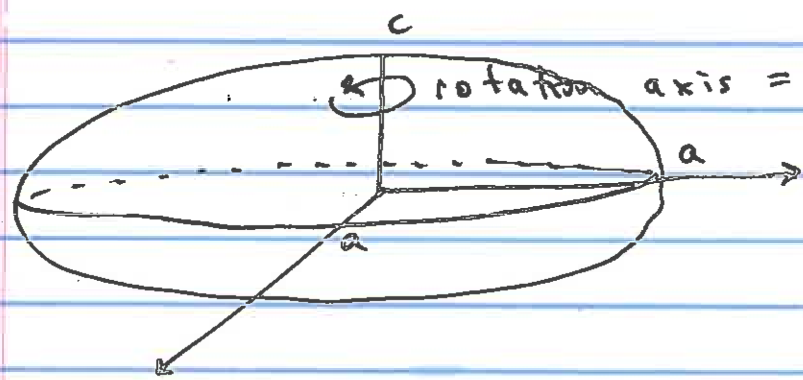
$\cos\theta(t)\,R(t) \equiv c_0 \equiv \cos\theta_0 \cdot R_0$

$\frac{\pi}{2} > \theta(t) > \theta_0$

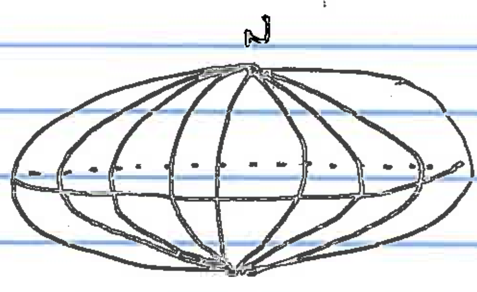
Caution $\left(\theta(t_1) = \frac{\pi}{2} \text{ for any } t_1 \implies \gamma \text{ is a meridian} \implies \theta(t) \equiv \frac{\pi}{2}\ \forall t \right)$

Ex Rotationally symmetric non-spherical Ellipsoid

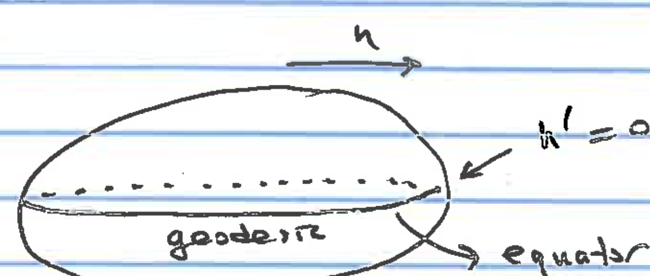
$$\underbrace{\frac{x^2}{a^2} + \frac{y^2}{a^2}}_{a=b} + \frac{z^2}{c^2} = 1.$$



All meridians are geodesics

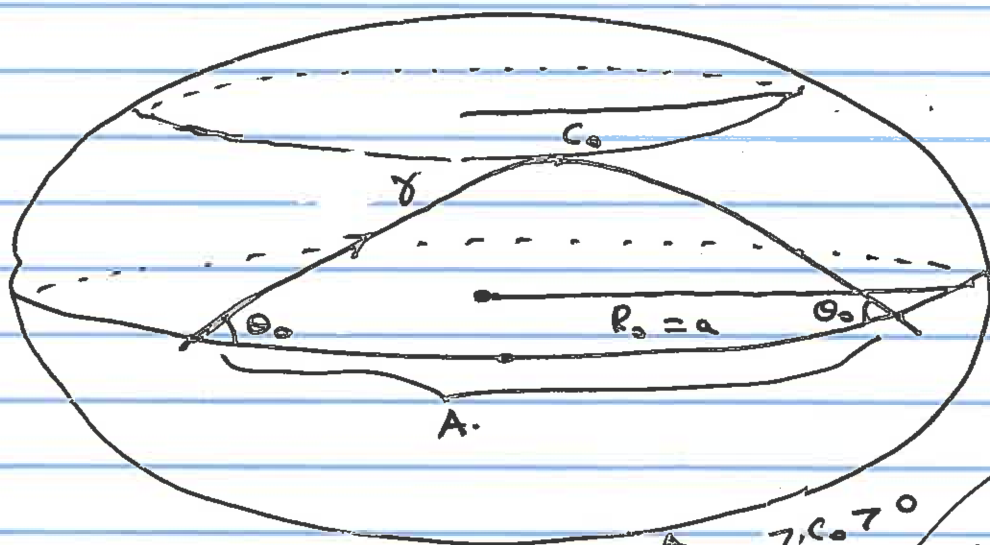


geodesics.



equator is the only geodesic which is a parallel.

Slanted geodesics of the ellipse $\frac{x^2+y^2}{a^2}+\frac{z^2}{c^2}=1.$



$a > C_0 > 0$

$C_0 = R_0 = a$
$\theta(t) \equiv 0$
equator

$C_0 = 0$
$\Rightarrow \theta(t) \equiv \frac{\pi}{2}$
$\Rightarrow$ meridian.

$0 \leq C_0 \leq \overset{R_0}{\overset{\shortparallel}{a}}$ $\qquad \cos\theta(t)\cdot R(t) \equiv C_0$

When $0 < C_0 < a$

① $\mathbb{Q} \ni \frac{2\pi R_0}{2A}$ (circumference of great circle / equator) (length of one cycle) 

② $\mathbb{Q} \not\ni \frac{2\pi R_0}{2A}$

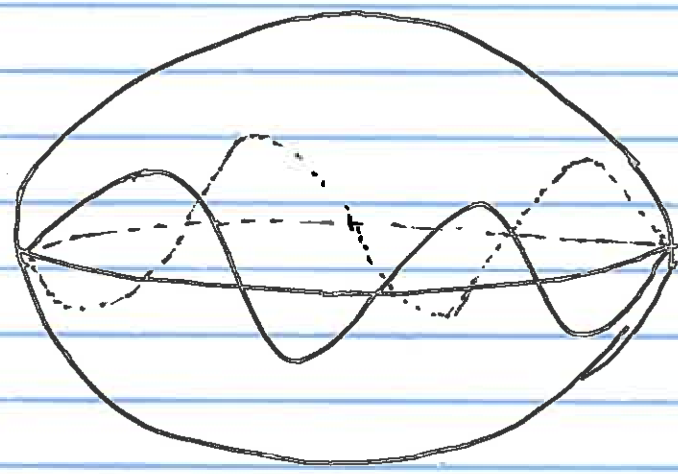
These give closed geodesic
Look at page ⑥

Non closed geodesic
Dense in a ribbon about equator

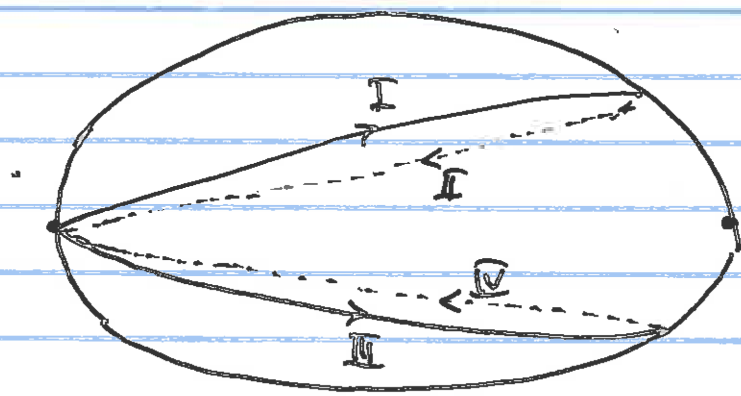
($\mathbb{Q}$: Rational numbers)

See p 219 colored pictures

Ellipse continue.



$$\frac{2\pi R_0}{2A} = 4$$



$$\frac{2\pi R_0}{2A} = \frac{1}{2}$$