1.3 **Defn** Let $\beta(s)$ be a regular curve parametrized w.r.t. arclength, $|\beta'(s)| \equiv 1$.
Define, the unit tangent vector: $T = \beta'(s)$.

Defn $|T'(s)| = \kappa(s)$ curvature of β at $\beta(s)$.

If $\kappa(s) \neq 0$, then $\dfrac{T'(s)}{\kappa(s)} = N(s)$ is called the principal normal.

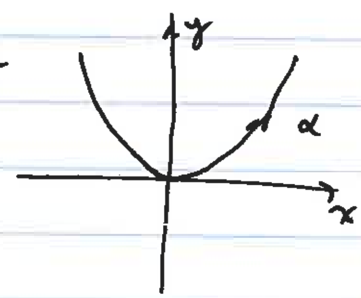
Caution: if $\kappa(s_0) = 0$, then $N(s_0)$ is undefined.

Obs 1) $|T'(s)| = |\beta''(s)| = \kappa(s)$

2) $T'(s) = \kappa(s) N(s)$

Q What do we do if the curve is not parametrized wrt arclength?

Ex $\alpha(t) = (t, t^2) : \mathbb{R} \to \mathbb{R}^2$



$\alpha'(t) = (1, 2t)$

$|\alpha'(t)| = \sqrt{1+4t^2} = \dfrac{ds}{dt}$

$s = \displaystyle\int_0^t \sqrt{1+4t^2}\, dt$

$\bar{\alpha}(s) = \alpha(t(s))$ be arclength parametrization.

$$\bar{T}(s) = \frac{d\bar{\alpha}}{ds} = \frac{d}{ds}\alpha(t(s)) = \frac{dt}{ds}\frac{d}{dt}\alpha(t)$$

$$= (1+4t^2)^{-\frac{1}{2}} \cdot \frac{d}{dt}(t, t^2)$$

$$= (1+4t^2)^{-\frac{1}{2}}(1, 2t) = \frac{(1, 2t)}{\|(1, 2t)\|} = T(t)$$

$$\kappa(s) N(s) = \frac{d\bar{T}}{ds} = \frac{d}{ds}(1+4t^2)^{-\frac{1}{2}} \cdot (1, 2t)$$

$$= \frac{dt}{ds}\frac{d}{dt}\left((1+4t^2)^{-\frac{1}{2}}(1, 2t)\right)$$

$$= (1+4t^2)^{-\frac{1}{2}}\left[-\frac{1}{2}(1+4t^2)^{-\frac{3}{2}} \cdot 8t \cdot (1, 2t) + (1+4t^2)^{-\frac{1}{2}} \cdot (0, 2)\right].$$

$$= (1+4t^2)^{-2}\left[-4t(1, 2t) + (1+4t^2)(0, 2)\right]$$

$$= (1+4t^2)^{-2}\left[(-4t + 0, -8t^2 + 2 + 8t^2)\right]$$

$$= (1+4t^2)^{-2}\left[(-4t, 2)\right] := \kappa(s) N(s)$$

$$\kappa(s) = \| (1+4t^2)^{-2} \cdot (-4t, 2) \|$$

$$= \frac{\sqrt{16t^2+4}}{(1+4t^2)^2} = 2(1+4t^2)^{-\frac{3}{2}}.$$

$$N(s) = \frac{(-4t, 2)}{\sqrt{4+16t^2}} = \frac{(-2t, 1)}{\sqrt{1+4t^2}}.$$

Let $\beta(s)$ be a regular curve in $\mathbb{R}^3$, $|\beta'(s)| \equiv 1$

Want To define $\left\{ \begin{array}{ll} B & \text{binormal} \\ \tau & \text{torsion.} \end{array} \right.$

If $\kappa(s_0) = 0$, then $N(s_0)$ is undefined,

$\Rightarrow B(s_0)$, $\tau(s_0)$ are undefined

<u>Recall</u> $\vec{u} \times \vec{v}$ in $\mathbb{R}^3$

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

$\vec{u} \times \vec{v} \perp u, v$ when $\neq 0$

$|\vec{u} \times \vec{v}| = |u||v| \cdot \sin\theta$ where $\theta = \measuredangle(\vec{u}, \vec{v})$

when $\vec{u}, \vec{v} \neq 0$

$u, v, u \times v$ satisfy RHR

right hand rule.

Summarize:

We are given $\beta(s)$, $\beta \in C^1$, $|\beta'(s)| = 1$.

$$\beta'(s) = T(s)$$

$$T'(s) \rightarrow |T'(s)| = \kappa(s)$$

when $\kappa(s) \neq 0$ $\quad \dfrac{T'(s)}{\kappa(s)} = N(s)$

$$|T| = 1$$
$$|N| = 1$$

$$T \cdot N = 0 \quad \Longleftarrow \quad \left[\begin{array}{l} |T| = 1 \\ T \cdot T = 1 \\ 2T' \cdot T = 0 \\ \kappa N = T' \\ 2\kappa N \cdot T = 0 \\ \kappa \neq 0 \\ N \cdot T = 0 \end{array}\right.$$

Defn For a given regular curve $\beta(s)$, $\beta \in C^1$, $|\beta'(s)| = 1$, when $\kappa(s) \neq 0$ one defines the binormal of β at $\beta(s)$

$$B(s) = T(s) \times N(s)$$

When $\kappa(s) \neq 0$: $|T| = |N| = |B| = 1$ $\quad |B| = |T||N| \cdot 1$

$$T \cdot N = T \cdot B = N \cdot B = 0 \qquad \angle\theta = \frac{\pi}{2}$$

$\{T(s), N(s), B(s)\}$ is an orthonormal basis of $\mathbb{R}^3$ at $\beta(s)$

$\{T(s), N(s), B(s)\}$ is called the moving frame or Frenet frame along the curve β.

<u>Caution</u> Need $\begin{cases} \beta \in C^1 \\ \beta \text{ regular} \\ \kappa > 0 \quad \text{at the point } \beta(s). \end{cases}$

We know how to find : $\beta \longrightarrow T, N, B, \kappa$. whenever $\kappa(s) > 0$

<u>Next to define torsion τ:</u>

<u>Lemma</u> $B'(s) \,//\, N(s)$ (when $\kappa(s) > 0$), unless $B'(s) = 0$

<u>Proof:</u> $B'(s) = a(s)\,T(s) + b(s)\,N(s) + c(s)\,B(s)$

$\{T, N, B\}$ orthonormal basis

$$a(s) = B'(s) \cdot T(s)$$
$$b(s) = B'(s) \cdot N(s)$$
$$c(s) = B'(s) \cdot B(s).$$

$B' = bN \Leftarrow \begin{cases} c(s) = 0 & \text{by } B \cdot B = 1 \Longrightarrow 2B \cdot B' = 0 \\ a(s) = 0 & \text{by the following} \end{cases}$

$$B = T \times N$$
$$B' = T' \times N + T \times N'$$
$$T' = \kappa N \Rightarrow \quad B' = \underbrace{\kappa N \times N}_{0} + T \times N'$$
$$B' = T \times N' \perp T$$
$$a(s) = B' \cdot T = 0$$

<u>Defn</u> Let β be a regular curve, $|\beta'(s)| \equiv 1$.
When $\kappa(s) \neq 0$, (T, B, N are defined)

We define the torsion of β at $\beta(s)$

to be $\tau(s) = -N(s) \cdot B'(s)$.

$$\left(\Longrightarrow \quad \underset{\text{since } B' /\!/ N}{B'(s) = -\tau(s) N(s)} \right)$$