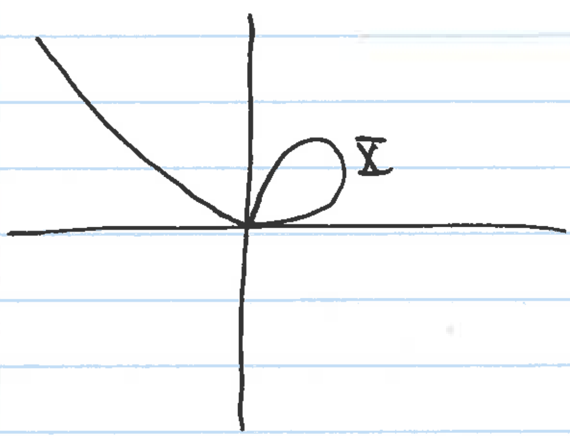
Aug 28, 2017

Question about an earlier Example.

Folium of Descartes

$$\alpha(t) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3} \right) : \underbrace{(-1,\infty)}_{I} \longrightarrow \mathbb{R}^2$$

Let $X = $ image of α $= \alpha((-1,\infty))$



$\alpha : I \longrightarrow X$

is 1-1, onto, continuous

since $\alpha : I \to \mathbb{R}^2$

is diff'ble.

<u>Claim</u> $\alpha^{-1} : X \longrightarrow I$

is <u>NOT</u> continuous.

Let $p_n = \alpha(n) = \left(\frac{3n}{1+n^3}, \frac{3n^2}{1+n^3} \right)$

As $n \to \infty$ $p_n \longrightarrow (0,0)$, but

$\lim_{n\to\infty} \alpha^{-1}(p_n) \neq \alpha^{-1}(0,0)$, since

$\alpha^{-1}(p_n) = n$, $\lim_{n\to\infty} \alpha^{-1}(p_n) = \lim_{n\to\infty} n = $ DNE.

$\alpha^{-1}(0,0) = 0$

<u>Recall Defn</u>: f is continuous at a if $\lim_{x\to a} f(x) = f(a)$.

we simply take $f = \alpha^{-1}$.

1.2

Defn For $\alpha : [a,b] \to \mathbb{R}^n$,

$$s(t) = \int_a^t |\alpha'(u)|\,du \quad \text{arclength function.}$$

Ex $\alpha(t) = \left(t, \sqrt{2}\ln t, -\frac{1}{t}\right) \qquad 1 \le t \le e$

$$\alpha'(t) = \left(1, \frac{\sqrt{2}}{t}, \frac{1}{t^2}\right)$$

$$|\alpha'(t)| = \sqrt{1 + \frac{2}{t^2} + \frac{1}{t^4}} = 1 + \frac{1}{t^2}$$

$$s(t) = \int_1^t \left(1 + \frac{1}{u^2}\right) du = u - \frac{1}{u}\Big|_1^t = t - \frac{1}{t}$$

$s(t) = t - \frac{1}{t}$ arclength function

$$s = t - \frac{1}{t} = \frac{t^2 - 1}{t}$$

$$st = t^2 - 1$$

$$0 = t^2 - st - 1$$

$$t = \frac{s \pm \sqrt{s^2+4}}{2} \geq 0 \qquad t = \frac{s + \sqrt{s^2+4}}{2}$$

parametrization wrt arclength:

$$\beta(s) = \alpha(t(s)) = \left(\frac{s+\sqrt{s^2+4}}{2}, \sqrt{2}\ln\frac{s+\sqrt{s^2+4}}{2}, -\frac{2}{s+\sqrt{s^2+4}}\right)$$

$|\beta'(s)| = 1$ so β has speed 1 Why? See p.3

Defn $\overset{u}{I} \xrightarrow{h} \overset{t}{J} \xrightarrow{\alpha} \mathbb{R}^n$

let α be a curve,

let h be an onto map from I to J

$\alpha(h(u)) = \beta(u)$ is called a reparametrization of α.

Obs $\beta'(u) = \alpha'(t) \cdot \frac{dt}{du} = \alpha'(t)\, h'(u)$

$t = h(u)$

Thm: If $\alpha : I \longrightarrow \mathbb{R}^n$ is regular, 1-1, then

α can be reparametrized by arclength via β (to have speed 1), so that β is C^1.

Proof: $\left.\begin{array}{l} \alpha'(t) \text{ exists} \\ |\alpha'(t)| > 0 \end{array}\right\}$ regular

$I = [a,b]$, $\quad s(t) = \int_a^t |\alpha'(u)|\,du.$

$$\frac{ds}{dt} = |\alpha'(t)| > 0$$

s is a strictly increasing function of t

s is 1-1.

$s : [a,b] \xrightarrow{\text{onto}} [c,d]$

Extreme Value thm, Intermediate Value Thm

$s(t)$ has an inverse $t = t(s)$.

$t(s)$ & $s(t) \in C^1$ (Inverse function Thm)

$$\alpha(t) = \alpha(t(s)) = \beta(s).$$

Claim: $|\beta'(s)| = 1$ proof:

$$\frac{d}{ds}\beta(s) = \frac{d}{ds}\alpha(t(s)) = \frac{d\alpha}{dt}\cdot\frac{dt}{ds}$$

$$|\beta'(s)| = \left|\frac{d\alpha}{dt}\right|\left|\frac{dt}{ds}\right| = |\alpha'(t)|\cdot\frac{1}{|\alpha'(t)|} = 1.$$

Corollary: If α, γ are 1-1, regular parametriz. of the same path, then α and γ are reparametrizations going in the same direction of each other.

Trick: reparametrize α wrt arclength: β_1
" γ " " : β_2

$\beta_1 = \beta_2$ since it is the same path going in the same direction.

$$\alpha \longrightarrow \beta_1 \xrightarrow{=} \beta_2 \longrightarrow \gamma.$$

successive reparametrizations.

1.3 Start

Lemma: Let $\alpha : I \longrightarrow \mathbb{R}^n$, $\beta : I \longrightarrow \mathbb{R}^n$ $\Big\}$ $\alpha, \beta \in C^1$.

Then

(i) $(\alpha \cdot \beta)' = \alpha' \cdot \beta + \alpha \cdot \beta'$

(ii) $\|\alpha\| = 1 \ \forall t \implies \alpha'(t) \perp \alpha(t)$

unless $\alpha'(t) = 0$

Caution $\longrightarrow$ ($\alpha \neq 0$ since $|\alpha| = 1$)

Proof: (i) $\alpha = (\alpha^1, \alpha^2, \dots \alpha^n)$

$\beta = (\beta^1, \beta^2, \dots \beta^n)$

$$\alpha \cdot \beta = \alpha^1 \beta^1 + \alpha^2 \beta^2 + \dots + \alpha^n \beta^n$$

$$(\alpha \cdot \beta)' = (\alpha^1)' \beta^1 + \alpha^1 (\beta^1)' + (\alpha^2)' \beta^2 + \alpha^2 (\beta^2)' + \vdots \ (\alpha^n)' \beta^n + \alpha^n (\beta^n)'$$

$$= (\alpha^1, \alpha^2, \dots \alpha^n)' \cdot (\beta^1 \dots \beta^n) + (\alpha^1, \dots, \alpha^n)(\beta^1, \dots \beta^n)'$$

$$= \alpha' \cdot \beta + \alpha \cdot \beta'.$$

(ii)

$$\alpha \cdot \alpha = |\alpha|^2 = 1$$

d/dt

$$\alpha' \cdot \alpha + \alpha \cdot \alpha' = 0$$

$$2\alpha' \cdot \alpha = 0$$

$\alpha \neq 0$ since $|\alpha| = 1$.

So: $\alpha'(t) = 0$

OR

$\alpha'(t) \perp \alpha(t)$