8/23/17

1

$f: D \subseteq \mathbb{R}^1 \to \mathbb{R}$

$C^k(D)$: Collection of all functions f s.t. all derivatives $f^{(i)}$ are continuous on D, $0 \leq i \leq k$. (i^{th} derivative of f)

$$C^k \subseteq \mathcal{D}^k \subseteq C^{k-1} \quad \forall k.$$

$\mathcal{D}^k$ is the collection of all functions f s.t. all derivatives $f^{(i)}$ exist on D for $0 \leq i \leq k$.

$$C^\infty = \bigcap_{k=0}^{\infty} C^k$$

e^x, polynomials, $\ln x$, $\sin x$, $\cos x$ $\in C^\infty$.

In general $\vec{f} = (f^1, f^2, \dots f^\ell) : I \subseteq \mathbb{R}^1 \to \mathbb{R}^\ell$

$\vec{f} \in C^k \iff$ Each component f^i belongs to C^k

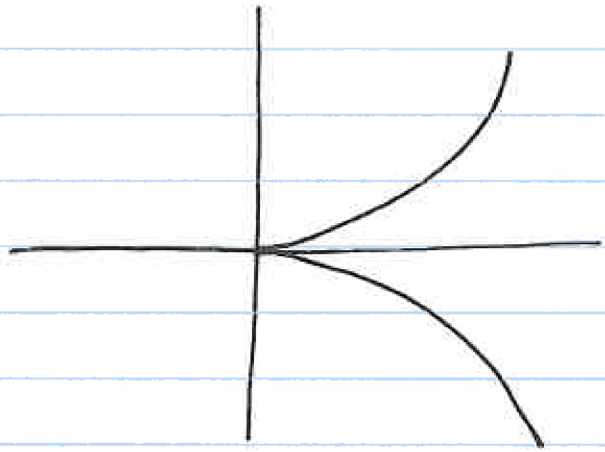
Defn A curve $\alpha : I \to \mathbb{R}^n$ is called regular if
(i) $\alpha \in C^1$
(ii) $\alpha' \neq \vec{0}$ on I.

Defn α' is called the velocity vector
$|\alpha'|$ is called the speed.

Ex $\quad \alpha(t) = (t^2, t^3) \qquad t \in \mathbb{R}.$

$$\left.\begin{aligned} x &= t^2 \\ y &= t^3 \end{aligned}\right\} \qquad x^3 = y^2$$

$$y = \pm\sqrt{x^3}$$

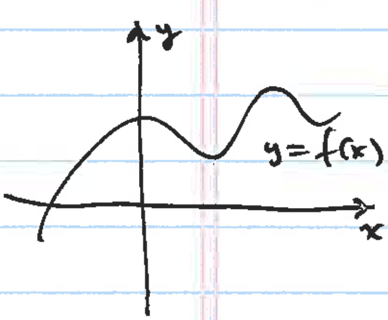


Cusp, at $(0,0)$

$$\left.\begin{aligned} &\alpha \in C^\infty \\ &\alpha' = (2t, 3t^2) \\ &\alpha'(0) = (0,0) \end{aligned}\right\} \quad \alpha \text{ is not } \underline{\text{regular on } \mathbb{R}.}$$

$$\left.\begin{aligned} &\alpha \text{ is regular on } (0,\infty) \\ &\alpha \text{ is } \quad \text{"} \quad \text{"} \quad (-\infty, 0) \end{aligned}\right\} \text{ but not on } \mathbb{R}$$

Ex



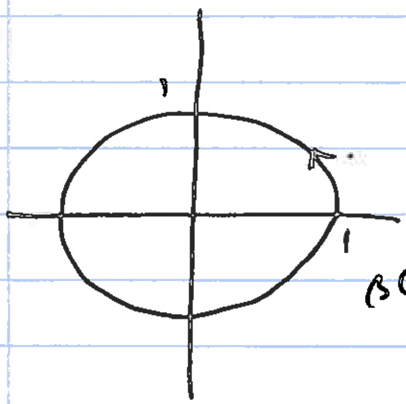
$\alpha(t) = (t, f(t))$ parametrizes the graph of $y = f(x)$

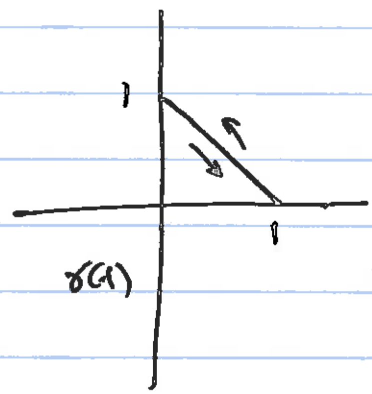
$$\alpha'(t) = (1, f'(t)) \neq (0,0)$$

α is regular if $f \in C^1$.

Ex

$$\left.\begin{aligned} \alpha(t) &= (\cos^3 t,\ \sin^3 t) \\ \beta(t) &= (\cos t,\ \sin t) \\ \gamma(t) &= (\cos^2 t,\ \sin^2 t) \end{aligned}\right\} \text{ all } \mathbb{R} \to \mathbb{R}^2$$

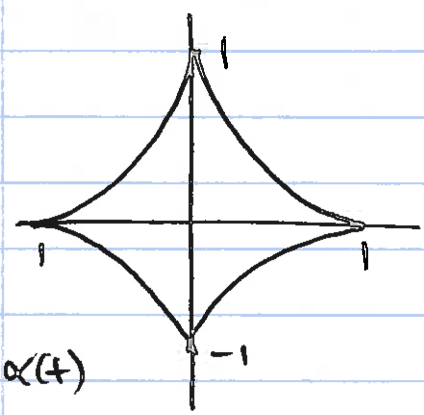




$x = \cos^2 t \geq 0$

$y = \sin^2 t \geq 0$

$x + y = 1$



$\cos^3 t = x$

$\sin^3 t = y$

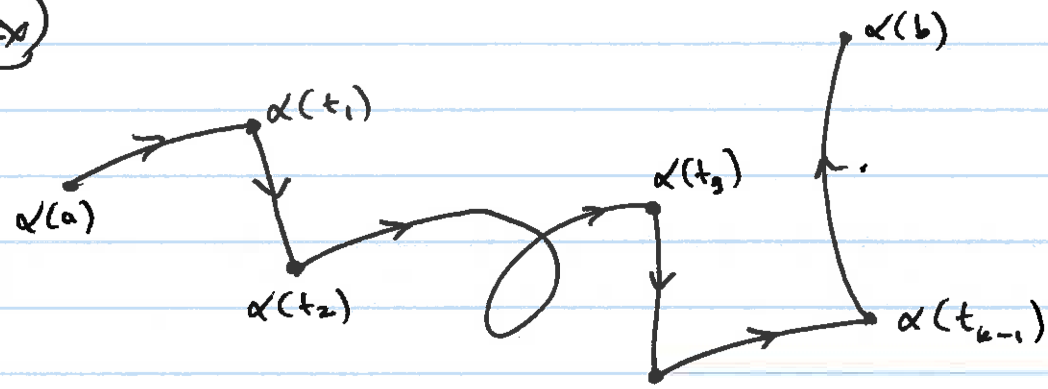
$$x^{2/3} + y^{2/3} = 1$$

Defn : A curve $\alpha : \underset{[a,b]}{\overset{\shortparallel}{I}} \to \mathbb{R}^n$ is called piecewise C^1, if $\exists$ a partition of $[a,b]$

$$a = t_0 < t_1 < t_2 < \cdots < t_k = b \quad \text{s.t.}$$

(restricted) $\alpha : [t_i, t_{i+1}] \to \mathbb{R}^n$ is a C^1 curve, and α is continuous on all of $[a,b]$.

Ex



Caution When we say $f : [c,d] \to \mathbb{R}$ is C^1, be careful

$$\left.\begin{aligned} f'(c) &= \lim_{x \to c^+} f'(x) \\ f'(d) &= \lim_{x \to d^-} f'(x) \end{aligned}\right\} \text{required}$$

Defn Let $\alpha : [a,b] \to \mathbb{R}^n$ be a piecewise C^1 curve. Then

pw

$$L(\alpha) = \text{length}(\alpha) = \int_a^b |\alpha'(t)|\, dt$$

Caution: This is a piecewise continuous function with jump discontinuities at $t = t_i$.

Caution α is piecewise C^1

$$L(\alpha) = \text{length}(\alpha) = \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} |\alpha'(t)|\, dt$$

Theorem: Let $\alpha : [a,b] \to \mathbb{R}^n$ be of class C^1.
(Actually true for pw C^1)

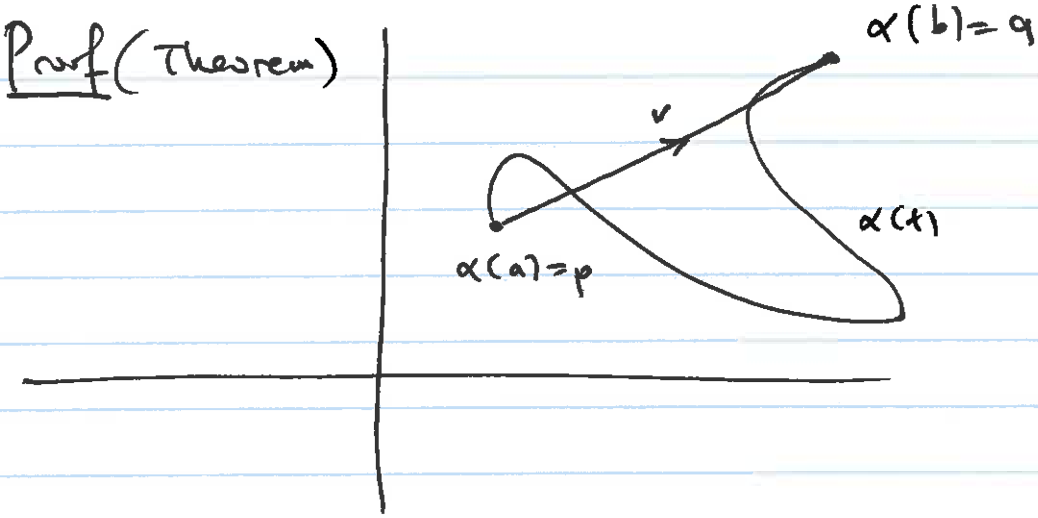
Then $\text{length}(\alpha) \geq \| \alpha(a) - \alpha(b) \|$.
Equality holds iff $\iff$ α traces the line segment from $\alpha(a)$ to $\alpha(b)$ going in one direction.

Recall Cauchy Schwartz ineq.

$$u \cdot v \leq \|u\| \|v\|$$

Also True: $u \cdot v = \|u\| \|v\| \cdot \cos\theta$ $\quad \theta = \measuredangle(u,v)$ when $u, v \neq 0$.

Proof (Theorem)



$\alpha(a) = p$

$\alpha(b) = q$ WLOG $p \neq q$.

$$v = q - p \neq 0$$

$$u = \frac{v}{|v|} \quad \text{unit}$$

$$\text{Length}(\alpha) = L(\alpha) = \int_a^b |\alpha'(t)|\,dt$$

1 $$\int_a^b \alpha'(t) \cdot u \, dt \leq \int_a^b |\alpha'(t)| \cdot \underbrace{|u|}_{1} \, dt = L(\alpha).$$

$\|$ u is constant vector

$$u \cdot \int_a^b \alpha'(t)\,dt \underset{\text{FTC}}{=} u \cdot \Big(\alpha(b) - \alpha(a)\Big) = u \cdot (q - p)$$

$$= u \cdot v = \frac{v}{|v|} \cdot v = \frac{|v|^2}{|v|} = |v| = |\alpha(b) - \alpha(a)|$$

Hence $L(\alpha) \geq |\alpha(b) - \alpha(a)|$.

What happens when

2 $L(\alpha) = |\alpha(b) - \alpha(a)|$ (WTS $\Rightarrow \alpha$ traces a line segment

$\Rightarrow$ all inequalities in 1, must be equalities

$$\int_a^b \alpha'(t)\cdot u \, dt = \int_a^b |\alpha'(t)|\cdot|u| \, dt$$

$f, g \in C^0$, $f \leq g$, $\int_a^b f\,ds = \int_a^b g\,ds$ $\Rightarrow f = g$.

$\Rightarrow \quad \alpha'(t)\cdot u = |\alpha'(t)|\cdot|u|$

(since $\alpha'(t)\cdot u \leq |\alpha'(t)|\cdot|u|$.

$\Rightarrow \quad \cos\theta = 1, \qquad \theta = \measuredangle(u, \alpha'(t)) = 0,$

unless $\alpha'(t) = 0$.

$\Rightarrow \quad \alpha'(t) \parallel u.$

$|\alpha'(t)|\cdot|u| = \alpha'(t)\cdot u = \lambda(t) \geq 0$

$\alpha'(t) = u\cdot\lambda(t)$

FTC: $\alpha(t) = \alpha(a) + \int_a^t \alpha'(s)\,ds = \alpha(a) + \int_a^t u\cdot\lambda(s)\,ds$

$$\alpha(t) = \alpha(a) + u\cdot \underbrace{\int_a^t \lambda(s)\,ds}_{\text{increasing function, since } \lambda \geq 0}$$

$\Rightarrow \alpha(t)$ is a line segment from p to q, tracing in one direction.