Aug 21, 2017

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MATH 4500

Oprea.

1.1 $I \subseteq \mathbb{R}$ always denotes an interval.

<u>Defn</u> A continuous function $\alpha : I \longrightarrow \mathbb{R}^n$ is called a curve.

<u>Notation</u> $\alpha(t) = (\alpha^1(t), \alpha^2(t), \alpha^3(t))$

α^i is <u>NOT</u> a power

α^i is the i <u>th</u> component

<u>Defn</u> A curve $\alpha(t)$ is called diff'ble if each α^i is diff'ble.

<u>$\mathcal{D}^i, C^k$ Notation</u>:

Let $f : I \to \mathbb{R}$

$C^0 = C^0(I)$ is the set of all continuous functions on I:

$f \in C^0(I) \iff \forall a \in I, \ \lim_{x \to a} f(x) = f(a)$, where all exist.

$\iff \forall a \in I \ \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall y \in I$

$|y - a| < \delta \implies |f(y) - f(a)| < \varepsilon$.

$\mathcal{D}' = \mathcal{D}'(I)$

f is diff'ble at a if $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists and is finite.

When this limit exists, we call it $f'(a)$.

$f \in \mathcal{D}' \iff f$ is diff'ble at each $a \in I$.

C^1: A function $f : I \subseteq \mathbb{R}^1 \to \mathbb{R}^1$ is called of class C^1 if $f'(x)$ exists on all of I, and $f'(x)$ is continuous on I.

$$C^1 \subseteq \mathcal{D}' \subseteq C^0$$

obvious from defn.

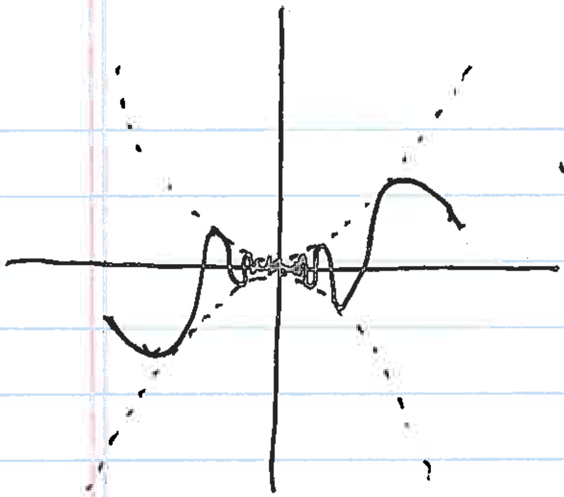
Prop: If f is diff'ble at a, then f is continuous at a.

$C^1 \neq \mathcal{D}'$, Example:

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} + x^2 \left(\cos \frac{1}{x}\right)\left(-\frac{1}{x^2}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

since:



$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h - 0} = \lim_{h\to 0} \frac{h^2 \sin\frac{1}{h} - 0}{h}$$

$$= \lim_{h\to 0} h \underbrace{\sin\frac{1}{h}}_{\text{bdd}} = 0$$

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \Big(\underbrace{2x \sin\frac{1}{x}}_{\downarrow\, 0} - \underbrace{\cos\frac{1}{x}}_{\substack{\text{oscillates} \\ \text{between } -1 \,\&\, 1}} \Big) = \text{DNE}$$

$$0 = f'(0) \neq \lim_{x\to 0} f'(x) = \text{DNE}.$$

$f'(x)$ is not continuous on $\mathbb{R}$.

$f(x) \notin C^1$, but $f \in \mathcal{D}^1$

Ex $\quad g(x) = |x| : \mathbb{R} \to \mathbb{R} \qquad g \in C^0$

$g \notin \mathcal{D}^1$

since $g'(0)$ DNE.