MATH 4210 Final Exam INFORMATION
Take home.

- The exam will be posted on ICON on Tuesday May 12 in the evening about 8 pm

- Exam is due 11:59 pm Thursday May 14, 2020 due on ICON for MATH 4210 0A01

Long Questions will be chosen from Chapters 5 & 7.

Chap 5 : Do 2 of 3 given questions
Chap 7 : Do 2 of 3 given questions
True/False : Do it all
For a total 5 questions.
If you answer more than 2 questions in either chap 5 or 7; The first 2 will be graded.

PLEASE
Either type or write legibly and large
Do the following for better scanned copies:

- Use White paper
- Use black ink, not pencil (doesn't scan well)
- Use every other line
- Write large,

An IMPORTANT APPLICATION (of $\mathscr{C}(X)$ complete)

Thm: Let $(\mathbb{X}, d)$ be a complete metric space.
$T: \mathbb{X} \to \mathbb{X}$ be such that, $\exists r$, $0 \le r < 1$
$d(T(x), T(y)) \le r\, d(x,y)$.
Then $\exists$ a unique $x_0 \in \mathbb{X}$ s.t. $T(x_0) = x_0$.

Known as Banach Fixed pt Thm, or Contraction Principle. (Rudin p 220)

Prof Uniqueness.
If $T(x_0) = x_0$ and $T(y_0) = y_0$, then
$d(x_0, y_0) = d(T(x_0), T(y_0)) \le r\, d(x_0, y_0)$
Which is not possible for $0 \le r < 1$ when $d(x_0, y_0) \ne 0$
So $x_0 = y_0$.

Existence Let $x_1 \in \mathbb{X}$ be an arbitrary pt.
Define $x_2 = T(x_1)$,
$x_{n+1} = T(x_n)$ $\forall n$.

Let $d(x_1, x_2) = A$. Claim $d(x_n, x_{n+1}) \le r^{n-1} A$
$n=1$ is $d(x_1, x_2) = A$.
$\forall n \ge 1$, $d(x_n, x_{n+1}) \le r^{n-1} A \Rightarrow$
$d(x_{n+1}, x_{n+2}) = d(T(x_n), T(x_{n+1}))$
$\le r\, d(x_n, x_{n+1}) \le r \cdot r^{n-1} A = r^n A$ #
proves the claim by induction

If $A = 0$, then x_1 is the fixed point: $f(x_1) = x_2 = x_1$
We may assume $A > 0$.

$\forall N \in \mathbb{N}\ \forall k \in \mathbb{N}$

$$d(x_N, x_{N+k}) \le d(x_N, x_{N+1}) + d(x_{N+1}, x_{N+2}) + \cdots + d(x_{N+k-1}, x_{N+k})$$

$$\le (r^{N-1} + r^N + \cdots + r^{N+k-2}) A$$
$$\le r^{N-1} A (1 + r + r^2 + \cdots r^{k-1})$$
$$\le r^{N-1} A \frac{1}{1-r}. \qquad (0 \le r < 1)$$

since $r^N \to 0$ for $0 \le r < 1$ $\Big\}$ $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\quad r^{N-1} < \frac{1-r}{A}\varepsilon$

Hence $\forall \varepsilon > 0\ \exists N\ \forall n, m \ge N,\ m > n$

$$d(x_n, x_m) \le r^{n-1} A \frac{1}{1-r} \le r^{N-1} A \frac{1}{1-r} < \varepsilon$$

Hence $\{x_n\}$ is Cauchy in X. X complete, so $\exists x_0 \in X,\ \lim_{n\to\infty} x_n = x_0$

Since $\forall \varepsilon > 0\ \exists \delta = \varepsilon$ s.t.

$$\forall x, y \in X \quad d(x,y) < \delta \Rightarrow d(T(x), T(y)) \le r\,d(x,y) < \varepsilon.$$

T is Continuous. $\lim_{n\to\infty} T(x_n) = T(\lim_{n\to\infty} x_n) = T(x_0)$

$$\lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x_0$$

$\Rightarrow T(x_0) = x_0$.

<u>PROP I</u> Let $t_0, y_0 \in \mathbb{R}$, and $\varepsilon, M, C, R > 0$ be s.t. $0 < \varepsilon < \min\left(\frac{1}{C}, \frac{R}{M}\right)$.

If $f(t,y) : D = I_t \times I_y \longrightarrow \mathbb{R}$
where
$$I_t = [t_0 - \varepsilon, t_0 + \varepsilon],\ I_y = [y_0 - R, y_0 + R]$$
satisfying that

(1) f is continuous on D, $\sup\limits_D |f(t,y)| \leq M$;

and

(2) f is uniformly Lipschitz in y with
$C > 0 : |f(t,y_1) - f(t,y_2)| \leq C|y_1 - y_2|$
$\forall t \in I_t \quad \forall y_1, y_2 \in I_y$

Then $$T(\varphi)(t) = y_0 + \int_{t_0}^{t} f(s, \varphi(s))\, ds$$

defines a transformation

$$T : \mathscr{C}_0 = \{\varphi : I_t \to I_y \mid \varphi \text{ continuous}\} \longrightarrow \mathscr{C}_0$$
satisfying

$$\|T(\varphi_1) - T(\varphi_2)\| \leq r \|\varphi_1 - \varphi_2\|$$
for $r = \varepsilon C < 1$.

<u>Remark:</u> f is continuously differentiable in a neighborhood of (t_0, y_0) $\Longrightarrow$ one finds M, C, R, then picks ε s.t. $0 < \varepsilon < \min\left(\frac{1}{C}, \frac{R}{M}\right)$ to satisfy the hypothesis.

FUNDAMENTAL THEOREM OF ODE (Picard-Lindelöf)

Let the initial value problem be given

$$\left.\begin{aligned} \frac{dy}{dt} &= f(y,t) \\ y(t_0) &= y_0 \end{aligned}\right\} (*)$$

where f is continuous and uniformly lipschitz in y. Then $\exists\, \varepsilon > 0$ s.t. $\exists$ a function $y(t) : (t_0 - \varepsilon, t_0 + \varepsilon) \longrightarrow \mathbb{R}$ satisfying $(*)$.

Proof of Prop I $\Longrightarrow$ FTODE

One finds M, C, R and picks $0 < \varepsilon < \min\left(\frac{1}{C}, \frac{R}{M}\right)$ to satisfy Prop I.

Define T: as in Prop I

$$T : \mathcal{C}_0 = \{ \varphi : I_t \to I_y \mid \varphi \text{ continuous and bounded} \} \to \mathcal{C}_0$$

$\mathcal{C}_0$ is complete, since $\mathcal{C}(I_t)$ is complete

Shown in Prop I

T is a contraction mapping with $0 \le \underset{\varepsilon C}{\overset{}{r}} < 1$. ($r = \varepsilon C$)

$\exists\, \varphi_0$ a fixed point of T: $T(\varphi_0) = \varphi_0$

$$\varphi_0(t) = T(\varphi_0)(t) = y_0 + \int_{t_0}^{t} f(s, \varphi_0(s))\, ds .$$ By FT Calculus:

$$\varphi_0'(t) = f(t, \varphi_0(t))$$

$$\varphi_0(t_0) = y_0 .$$

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Proof Prop I. Let $\varphi : I_t \to I_r$ be continuous

① $$|(T\varphi)(t) - y_0| = \left| \int_{t_0}^{t} f(s, \varphi(s))\, ds \right|$$
$$\leq \left| \int_{t_0}^{t} |f(s, \varphi(s)|\, ds \right| \leq \left| \int_{t_0}^{t} M\, ds \right| \leq (t - t_0) M \leq \varepsilon M \leq R.$$

$\varphi \in \mathcal{C}_0 \implies T\varphi \in \mathcal{C}_0$, since integral of a continuous function is also continuous, and $T\varphi(t) \in [y_0 - R, y_0 + R] \in I_y$.

②

For $\varphi_1, \varphi_2 \in \mathcal{C}_0$

$$|(T\varphi_1)(t) - (T\varphi_2)(t)| = \left| \int_{t_0}^{t} f(s, \varphi_1(s)) - f(s, \varphi_2(s))\, ds \right|$$
$$\leq \left| \int_{t_0}^{t} |f(s, \varphi_1(s)) - f(s, \varphi_2(s))|\, ds \right|$$
$$\leq \left| \int_{t_0}^{t} C\, |\varphi_1(s) - \varphi_2(s)|\, ds \right|$$
$$\leq \left| \int_{t_0}^{t} C\, \|\varphi_1 - \varphi_2\|\, ds \right| \leq C\, |t - t_0|\, \|\varphi_1 - \varphi_2\|$$
$$\leq \underbrace{C\varepsilon}_{r} \|\varphi_1 - \varphi_2\|$$
$$\leq r \|\varphi_1 - \varphi_2\|$$

$$\|(T\varphi_1) - T\varphi_2\| = \sup_{t \in I_t} |T\varphi_1(t) - T\varphi_2(t)| \leq r\, \|\varphi_1 - \varphi_2\|.$$

$r = \varepsilon C < 1$ since $\varepsilon < \min\left(\frac{1}{C}, \frac{R}{M}\right)$