EQUICONTINUITY & COMPACTNESS in $\mathscr{C}(X)$

OVERALL VIEW

Chap II In metric spaces
Compact $\Longrightarrow$ closed and bounded.
$\not\Longleftarrow$ (Q)

Heine Borel **Thm** in $\mathbb{R}^n$
Compact $\Longleftrightarrow$ closed & bounded

Thm: In Every metric space the following are equivalent
(a) K is compact
(b) Every infinite subset $E \subseteq K$, $E' \cap K \neq \phi$
(c) Every sequence in K has a convergent subsequence whose limit is in K.

In $\mathscr{C}(K)$: where K is compact

ARZELA-ASCOLI w/ 7.23

Thm 7.25 If K is compact, $\forall n \in \mathbb{N}$, $f_n \in \mathscr{C}(K)$,
The family $\{f_n\}$ is pointwise bounded and equicontinuous, then
(i) $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded on K, and
(ii) $\{f_n\}_{n=1}^{\infty}$ contains a uniformly convergent subsequence.

Thm 7.24 If K is compact, $\forall n$ $f_n \in \mathscr{C}(K)$, then
$f_n \to f$ uniformly $\Rightarrow \{f_n\}$ is equicontinuous

Exc 7.19 Let K be a compact metric space.
Let $S \subseteq (\mathscr{C}(K), d_{\mathscr{C}})$.

S is compact in $\mathscr{C}(K)$
$\Longleftrightarrow$
S is uniformly closed, pointwise bounded and equicontinuous

***** Theorem 7.25

Let K be compact, $\forall n \in \mathbb{N}$, $f_n \in \mathscr{C}(K)$.
Let $\{f_n\}_{n=1}^{\infty}$ be pointwise bounded and equicontinuous. Then

(i) $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded on K, and

(ii) $\{f_n\}_{n=1}^{\infty}$ contains a uniformly convergent subsequence.

Proof K compact
$\forall f_n \in \mathscr{C}(K)$, $\{f_n\}$ equicontinuous

Let $\varepsilon > 0$ $\exists \delta > 0$
$\forall x, y \in K$ $\forall n \in \mathbb{N}$ $d(x,y) < \delta \Rightarrow |f_n(x) - f_n(y)| < \varepsilon$

Cover K, $\bigcup_{p \in K} N_\delta(p) \supseteq K$ compact

$\Rightarrow \exists$ a finite subcover $K \subseteq \bigcup_{i=1}^{\ell} N_\delta(p_i)$

$\{f_n\}$ pointwise bounded at each p_i

$\sup_{n \in \mathbb{N}} |f_n(p_i)| = M_i < \infty$

Let $M = \max(M_1, M_2, .. M_\ell) + \varepsilon$

$\forall x \in K \subseteq \bigcup_{i=1}^{\ell} N_\delta(p_i)$, $x \in N_\delta(p_{i_0})$ for some i_0.

$d(x, p_{i_0}) < \delta \Rightarrow \forall n \ |f_n(x) - f_n(p_{i_0})| < \varepsilon$

$|f_n(x)| \leq |f_n(p_{i_0})| + \varepsilon \leq M_{i_0} + \varepsilon \leq M.$

(i)✓ $\forall x \in K\ \forall n \in \mathbb{N}$, $|f_n(x)| \leq M$ uniformly bounded

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(ii) If K is compact, then $\exists$ a countable E s.t. $\bar{E} = K$ (Exc. 2.25)

Recall Thm 7.23 $\exists$ a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ s.t. f_{n_k} converges on E.

$\forall i$ Set $g_i = f_{n_i}$ (to shorten writing.)

$\{f_n\}$ equicontinuous $\Rightarrow \{g_i\}$ equicontinuous:

Let $\varepsilon > 0$ be given, $\exists \delta > 0\ \forall x, y \in K$

$\forall n \in \mathbb{N}\ \ d(x,y) < \delta \Rightarrow |f_n(x) - f_n(y)| < \varepsilon$

$\Rightarrow \forall i \in \mathbb{N}\ \ d(x,y) < \delta \Rightarrow |g_i(x) - g_i(y)| < \varepsilon.$ (1)

Know: g_i converges on E (2)

Want: g_i converges uniformly on K.

$\forall p \in K$, $N_\delta(p) \cap E \neq \emptyset$ since $\bar{E} = K$.

For any $x \in N_\delta(p) \cap E$, $p \in N_\delta(x)$

$\bigcup_{x \in E} N_\delta(x) \supseteq K$, compact $\Rightarrow K \subseteq \bigcup_{s=1}^{m} N_\delta(x_s)$ (3)

for some $x_s \in E$. $s = 1, .. m$

— $\forall x_s \quad s=1,\dots m \quad \lim_{i\to\infty} g_i(x_s)$ exists by ②

$g_i = f_{n_i}$

convergent $\Rightarrow$ Cauchy

$\forall s \quad \exists N_s \quad \forall i,j \in \mathbb{N} \quad i,j \geq N_s \quad |g_i(x_s) - g_j(x_s)| < \varepsilon$

Let $N = \max(N_1, \dots, N_s, \dots N_m)$

$\forall i,j \geq N \quad \forall s=1,\dots m$

$$|g_i(x_s) - g_j(x_s)| < \varepsilon \qquad ④$$

We want UCCUC for $g_i(x)$.

Let $x \in K$, $\quad x \in N_\delta(x_s)$ for some s, by ③

$$d(x,x_s) < \delta \Rightarrow |g_i(x) - g_i(x_s)| < \varepsilon \qquad ⑤$$

by equicontinuity by ①

$\forall i,j \geq N \quad |g_i(x) - g_j(x)|$

$$\leq \underbrace{|g_i(x) - g_i(x_s)|}_{\substack{<\varepsilon \\ \text{by } ⑤}} + \underbrace{|g_i(x_s) - g_j(x_s)|}_{\substack{<\varepsilon \\ \text{by } ④}} + \underbrace{|g_j(x_s) - g_j(x)|}_{<\varepsilon \text{ by } ⑤}$$

We have established $\forall \varepsilon > 0 \ \exists N$ (as in ④)

$\forall i,j \geq N \quad \forall x \in K \quad |g_i(x) - g_j(x)| < 3\varepsilon$

$\{g_i\}$ is uniformly Cauchy.

$\mathcal{C}(K)$ is complete, wrt $\|f-g\|$ metric

$g_i = f_{n_i}$ converges uniformly on K.

<u>EXAMPLE</u> Given $M, N \in \mathbb{R}^+$,
K a compact metric space, $x_0 \in K$

$$S = \{ f : K \to \mathbb{R} \mid \forall x,y \in K \quad |f(x) - f(y)| \le M\, d(x,y) \text{ and } |f(x_0)| \le N \}$$

S is compact wrt $d_{\mathscr{C}}$. in $\mathscr{C}(K)$

- Equicontinuous: $\forall \varepsilon > 0 \; \exists \delta = \frac{\varepsilon}{M}$
  $\forall x,y \in K \quad d(x,y) < \delta \Rightarrow$
  $|f(x) - f(y)| \le M\, d(x,y) < M\delta = \varepsilon$

- uniformly bounded:
  $d(x_0, x) : K \longrightarrow \mathbb{R}$ continuous.
  K compact $\Rightarrow \exists y_0 \in K \quad d(x_0, y_0) \ge d(x_0, y)$ for all $y \in K$.

  $\forall f \in S \; \forall y \in K$

  $$\begin{aligned} |f(y)| &\le |f(y) - f(x_0)| + |f(x_0)| \\ &\le M\, d(y, x_0) + N \\ &\le M\, d(x_0, y_0) + N \end{aligned}$$ independent of y and f.

Hence by Thm 7.25 every sequence $\{f_n\}$
uniformly convergent subsequence, $\{f_{n_k}\}$.

$\forall x,y \in K \quad |f_{n_k}(x) - f_{n_k}(y)| \le M\, d(x,y)$.

$$f_{n_k} \to f \text{ uniformly}$$

$$-M d(x,y) \le f_{n_k}(x) - f_{n_k}(y) \le M d(x,y)$$

$$\downarrow \quad \downarrow$$

$$-M d(x,y) \le f(x) - f(y) \le M d(x,y)$$

$$|f(x) - f(y)| \le M d(x,y).$$

Also $f_{n_k}(x_0) \to f(x_0)$; $|f_{n_k}(x_0)| \le N \Rightarrow |f(x_0)| \le N$

So $f \in S$.

S is closed, by same reasoning
S is compact by Thm. 7.25.

We mentioned that for diff'ble f on $[a,b] = D$

$$\forall x \in D \ |f'(x)| \le M \iff \forall x, y \in D, \ |f(x) - f(y)| \le M|x-y|$$

Caution

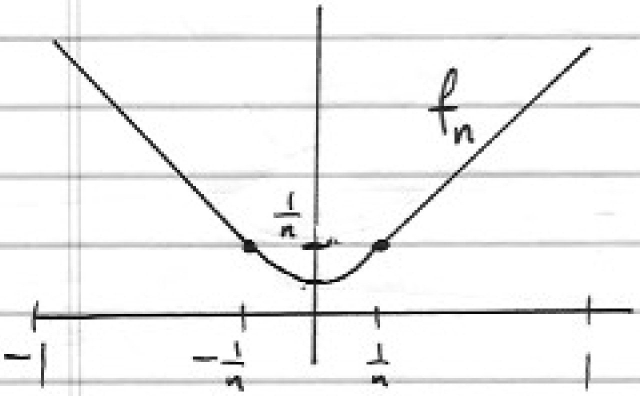
$$\begin{aligned} A &= \{ f : D \to \mathbb{R} \mid f \text{ diff'ble}, \ \|f'(x)\| \le M \} \\ &= \{ f : D \to \mathbb{R} \mid f \text{ diff'ble}, \ |f(x) - f(y)| \le M|x-y| \} \end{aligned}$$

is <u>not</u> uniformly closed.

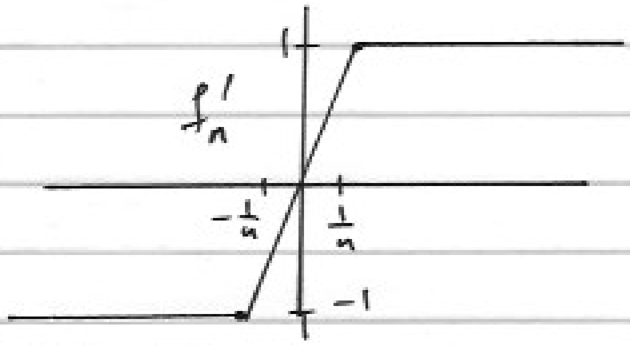
PTO for an example.

Ex

$$f_n(x) = \begin{cases} |x| & \text{if } |x| \geq \frac{1}{n}. \\ \frac{n}{2}x^2 + \frac{1}{2n} & \text{if } |x| < \frac{1}{n}. \end{cases} \quad : [-1,1] \to \mathbb{R}$$



$$f_n'(x) = \begin{cases} 1 & \text{if } x \geq \frac{1}{n} \\ -1 & \text{if } x \leq -\frac{1}{n} \\ nx & \text{if } |x| < \frac{1}{n} \end{cases}$$



$f_n(x) \longrightarrow |x|$ uniformly. $\|f_n - f\| = \frac{1}{2n}$

$|f_n(x) - f_n(y)| \leq |x-y|$

$f_n \in A$, $|x| \notin A$.

f_n are continuously diff'ble
f is not diff'ble at 0

$\lim_{n\to\infty} f_n' = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$, If you recall Thm 5.12,

This function is not the derivative of any function, not even $|x|$.