COMPACTNESS IN $\mathscr{C}(X)$ and EQUICONTINUITY

<u>Defn</u> A family of functions $\mathcal{F} = \{f_\alpha : E \to \mathbb{R} \mid \alpha \in \Lambda\}$ is called

- Pointwise bounded if $\forall x \in E \ \exists M_x \in \mathbb{R}$ s.t. $\forall f \in \mathcal{F}, \ |f(x)| \leq M_x$.

- Uniformly bounded if $\exists M \in \mathbb{R}$ s.t. $\forall f \in \mathcal{F}, \forall x \in E \quad |f(x)| \leq M$

- Equicontinuous if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall f \in \mathcal{F}, \ \forall x, y \in E, \ d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

<u>Ex</u> $\mathcal{F} = \{ f(x) = mx + b : [0,1] \to \mathbb{R} \mid m, b \in \mathbb{R} \}$

Given $A \in \mathbb{R}^+$ $\mathcal{F}_A = \left\{ f(x) = mx+b : [0,1] \to \mathbb{R} \;\middle|\; \begin{array}{l} |m| \leq A \\ b \in \mathbb{R} \end{array} \right\}$

Given $A, B \in \mathbb{R}^+$ $\mathcal{F}_{A,B} = \left\{ f(x) = mx+b : [0,1] \to \mathbb{R} \;\middle|\; \begin{array}{l} |m| \leq A \\ |b| \leq B \end{array} \right\}$

<u>Obs</u> $\forall f = mx+b : [0,1] \longrightarrow \mathbb{R}, \ \forall \varepsilon > 0 \ \exists \delta = \frac{\varepsilon}{|m|+1} > 0$

$\forall x, y \in [0,1], \ d(x,y) < \delta \Longrightarrow$

$$|f(x) - f(y)| \leq |(mx+b) - (my+b)| \leq |m||y-x|$$

$$< \delta |m| \leq \frac{|m|}{|m|+1} \varepsilon < \varepsilon.$$

$F_{A,B}$ · is uniformly bounded by $A+B$.

· is equicontinuous

F_A · Equicontinuous
· not uniformly bounded
· not pointwise bounded

F : equicontinuity fails.

$$|f(x)-f(y)| = |m||x-y|, \quad m \in \mathbb{R}$$

$|m|$ unbounded

$$\delta = \frac{\varepsilon}{|m|+1}$$

as $|m|$ gets smaller, we need smaller δ for the same ε.

Formally

need: not $(\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall f \in F \; \forall x, y \in E$
$d(x,y) < \delta \Rightarrow |f(x)-f(y)| < \varepsilon)$

need: $\exists \; \varepsilon = 1 \; \forall \delta = \frac{1}{n} \; (n \in \mathbb{N}) \; \exists f_n \in F \; \exists x_n, y_n \in E$
$d(x_n, y_n) < \frac{1}{n}$ but
$|f_n(x_n) - f_n(y_n)| \geq 1$

$\forall \delta = \frac{1}{n} \quad f_n = 4nx : [0,1] \to \mathbb{R}$

$\exists \; x_n = 0$
$y_n = \frac{1}{2n}$ $\quad |x_n - y_n| \leq \frac{1}{2n} < \frac{1}{n}$

but $|f_n(x_n) - f_n(y_n)| = |0 - 4n \cdot \frac{1}{2n}| = 2 > 1.$

Observe

Equicontinuous $\Longrightarrow$ Uniformly continuous.

$\not\Longleftarrow$

Ex. $\mathcal{F}'$ above.

Recall Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.
(Bolzano–Weierstrass)

Given (X, d) metric space

$$\mathcal{C}(X) = \{ f : X \to \mathbb{R} \mid f \text{ bounded \& continuous} \}$$

$$d_{\mathcal{C}}(f,g) = \|f-g\| = \sup_{x \in X} |f(x) - g(x)|$$

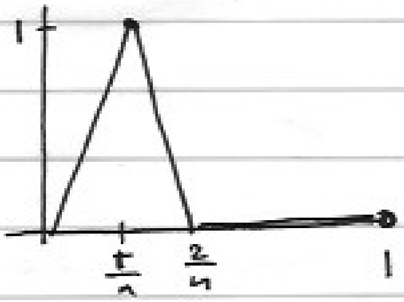
Thm: $\mathcal{C}(X)$ is a complete metric space

Question Does every bounded sequence in $\mathcal{C}(X)$ have a uniformly convergent subsequence?

ANSWER: NO in general. See the next page

Thm 7.25 If X is compact, and $\{f_n\}$ is a sequence which is equicontinuous and pointwise bounded on X, then $\exists$ a convergent subsequence $f_{n_k} \to f$ uniformly.

Example (Easier than 7.20)



$$f_n(x) = \begin{cases} nx & \text{if } 0 \le x \le \frac{1}{n} \\ n\left(\frac{2}{n} - x\right) & \text{if } \frac{1}{n} < x \le \frac{2}{n} \\ 0 & \text{if } x > 2/n. \end{cases}$$

$\{f_n\}$ is uniformly bounded: $\forall x \; \forall n \; |f_n(x)| \le 1$

Given $x \in (0,1]$ $\exists n \in \mathbb{N}$ $\frac{2}{n} < x$, $\forall m \geq n$
$\Rightarrow f_m(x) = 0$

$\forall x \in (0,1]$, $\lim_{n\to\infty} f_n(x) = 0$

$$\lim_{n\to\infty} f_n(0) = 0$$

$f_n(x) \longrightarrow f(x)$ pointwise, $f(x) \equiv 0$ on $[0,1]$

$\forall n$ $\|f_n - f\| = \sup_{x\in[0,1]} |f_n(x) - 0| = 1.$

Recall $f_n \longrightarrow f$ uniformly
$\iff \|f - f_n\| = \varepsilon_n \to 0.$

For NO subsequence n_k of n,

f_{n_k} can converge to f uniformly

There are several steps to prove Thm 7.25

STEP I:

***** **Theorem 7.23**

Let E be a countable set.
Let $\forall n$ $f_n : E \to \mathbb{R}$ be pointwise bdd. $(*)$
Then
$\{f_n\}$ has a subsequence $\{f_{n_k}\}$ s.t.
$f_{n_k}(x)$ converges $\forall x \in E$, as $k \to \infty$.

Proof Let $\{x_i \mid i \in \mathbb{N}\} = E$ be an ordering of E.

Start with x_1.

$\{f_n(x_1)\}$ is a sequence of real numbers.
it is bounded $(*)$, hypothesis.
Bolzano-Weierstrass for $\mathbb{R}$ tells us that

$\{f_n(x_1)\}$ has a convergent subsequence:

$$S_1 : f_{1,1}(x_1), f_{1,2}(x_1), f_{1,3}(x_1), \dots \longrightarrow A_1 \in \mathbb{R}.$$

next plug in x_2
$\{f_{1,n}(x_2)\}$ is a bounded sequence in $\mathbb{R}$.
So it has a convergent subseq.

$$S_2 : f_{2,1}(x_2), f_{2,2}(x_2), f_{2,3}(x_2), \dots \longrightarrow A_2 \in \mathbb{R}.$$

$$\lim_{n\to\infty} f_{2,n}(x_1) = A_1$$ since $\{f_{2,n}\}$ is a subseq of $\{f_{1,n}\}$

$$\lim_{n\to\infty} f_{2,n}(x_2) = A_2$$

S_1: $f_{1,1}$ $f_{1,2}$ $f_{1,3}$ ... $f_{1,n}$ .. converge at x_1 to A_1

S_2 $f_{2,1}$ $f_{2,2}$ $f_{2,3}$ .. $f_{2,n}$ .. " at x_2 to A_2

⋮

S_n $f_{n,1}$ $f_{n,2}$ $f_{n,3}$ $f_{n,n}$ " at x_n to A_n

$\forall n \geq 2$ · Each S_n is a subsequence of S_{n-1}

$\{f_{n-1,k}(x_n)\}_{k=1}^{\infty}$ bounded sequence in $\mathbb{R}$

$\{f_{n,k}(x_n)\}_{k=1}^{\infty}$ convergent sequence in $\mathbb{R}$

$$\lim_{k\to\infty} f_{n,k}(x_n) = A_n$$

$$\lim_{k\to\infty} f_{n,k}(x_i) = A_i \qquad 1 \leq i \leq n$$

Take $S: f_{1,1}\ f_{2,2}\ f_{3,3} \cdots f_{nn} \cdots$

S is a subsequence of all S_n

except possibly the first $(n-1)$ terms

since $S_{\ell\ell}$ might be removed later.

$\forall n \in \mathbb{N} \quad \lim_{k\to\infty} f_{k,k}(x_n) = \lim_{k\to\infty} f_{n,k}(x_n) = A_n$

$\{f_{k,k}\}_{k=1}^{\infty}$ is a subsequence of $\{f_n\}$

we want, that converges on all $E = \{x_i \mid i \in \mathbb{N}\}$

#

<u>Example</u> Let $M \in \mathbb{R}^+$ be given

$$A = \{ f : [a,b] \to \mathbb{R} \mid f \text{ diff'ble}, \ |f'(x)| \le M \}$$

$$B = \{ f : [a,b] \to \mathbb{R} \mid f \text{ diff'ble} \ \& \ |f(x) - f(y)| \le M|x-y| \}$$

Observe that $A = B$.

$\cdot \ \forall x \neq y \in [a,b] \ \exists z$ between x & y s.t. MVT

$$M \geq |f'(z)| = \frac{|f(x) - f(y)|}{|x-y|}$$

So $A \subseteq B$.

$$M \geq \lim_{x \to y} \frac{|f(x) - f(y)|}{|x-y|} = |f'(y)| \ ; \ \text{so } B \subseteq A.$$

$M = 1$
$|x| \in C$
$|x| \notin A = B$

$$A = B \subsetneq C = \{ f : [a,b] \to \mathbb{R} \mid |f(x) - f(y)| \le M|x-y| \}$$

C is an equicontinuous family

$\forall \varepsilon > 0 \ \exists \delta = \frac{\varepsilon}{M} \ \forall x, y \in [a,b] \quad |x-y| < \delta \Rightarrow$

$$|f(x) - f(y)| \le M|x-y| < M\delta = \varepsilon$$

EQUICONTINUITY & COMPACTNESS in $\mathscr{C}(X)$

OVERALL VIEW

Chap II . In Metric spaces

· Compact $\Longrightarrow$ closed & bounded. (2.34, Notes)

· $\not\Longleftarrow$

(2.41) Thm in $\mathbb{R}^n$ compact $\Longleftrightarrow$ closed & bounded (Heine-Borel)

2.37 + p40 Thm: · In every metric space the following are equivalent

(a) · K · compact

(b) · Every infinite subset $E \subseteq K$, $E' \neq \phi$

(c) · Every sequence in K has a convergent subsequence whose limit in K.

ARZELA-ASCOLI Thm. Thm: Let K be a compact metric space.

$\mathscr{C}(K) = \{ f : K \to \mathbb{R} \mid f \text{ continuous} \}$

$d_{\mathscr{C}}(f,g) = \sup_{x \in K} |f(x) - g(x)| = \|f-g\|$

Then

A subset $\mathcal{F} \subseteq \mathscr{C}(K)$ is compact if and only if $\mathcal{F}$ is closed, bounded, and equicontinuous.

We prove:

Thm 7.25: Let K be compact, $\forall n \in \mathbb{N}$, $f_n \in \mathscr{C}(K)$

Let $\{f_n\}_{n=1}^{\infty}$ be pointwise bounded and equicontinuous.

Then (i) $\{f_n\}_{n=1}^{\infty}$ Uniformly bounded

(ii) $\{f_n\}_{n=1}^{\infty}$ contains a uniformly convergent subsequence.