May 1, 2020

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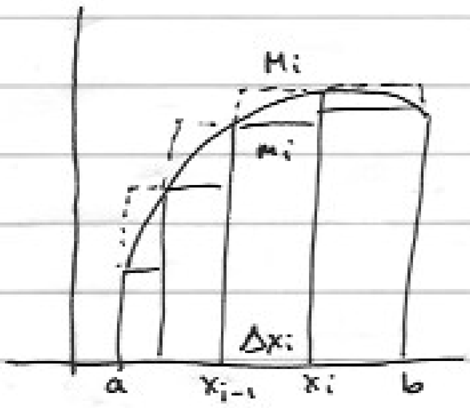
Basic Riemann Integral Review

Let $f : [a,b] \to \mathbb{R}$ be bounded.

Partition: $P = \{x_0, x_1, \ldots x_n\}$ $\quad a \le x_0 < x_1 < \ldots < x_n = b$

$\Delta x_i = x_i - x_{i-1}$



$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$$

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$$

$$\|P\| = \max_{1 \le i \le n} |x_i - x_{i-1}|$$

$U(P,f) = \sum M_i \Delta x_i$ $\qquad M = \sup_{[a,b]} f(x) < \infty$

$L(P,f) = \sum m_i \Delta x_i$ $\qquad -\infty < m = \inf_{[a,b]} f(x)$

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$$

$$\left.\begin{aligned} \underline{\int_a^b} f\,dx &= \sup_P L(P,f). \\ \overline{\int_a^b} f\,dx &= \inf_P U(P,f) \end{aligned}\right\} \text{ both exist.}$$

A function f is called Riemann integrable if

$\underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx$. We write $f \in \mathcal{R}$, in that case.

If $f \in \mathcal{R}$, $\int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$.

Chap 7
Uniform Convergence & Integration.

Thm 7.16 Let $\forall n \in \mathbb{N}$, $f_n : [a,b] \to \mathbb{R}$ bdd.
$f_n \to f$ uniformly. on $[a,b]$.

$$\forall n \; f_n \in \mathcal{R} \Rightarrow f \in \mathcal{R} \text{ and}$$

$$\int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx$$

Proof:

Let $\varepsilon_n = \|f_n - f\| = \sup_{x\in[a,b]} |f_n(x) - f(x)|$

$f_n \to f$ uniformly $\Rightarrow \varepsilon_n \to 0$.

$\forall x \in [a,b] \quad |f_n(x) - f(x)| \le \varepsilon_n$

$\forall x \in [a,b] \quad -\varepsilon_n \le f_n(x) - f(x) \le \varepsilon_n$

$\forall x \in [a,b] \quad f_n(x) - \varepsilon_n \le f(x) \le f_n(x) + \varepsilon_n$

$$\underline{\int_a^b} (f_n(x) - \varepsilon_n)\,dx \le \underline{\int_a^b} f\,dx \le \overline{\int_a^b} f\,dx \le \overline{\int_a^b} (f_n(x) + \varepsilon_n)\,dx$$

$\| \; f_n \in \mathcal{R}$ $\qquad\qquad \| \; f_n \in \mathcal{R}$

$\int_a^b f_n(x)\,dx - (b-a)\varepsilon_n \leftarrow$ differ by $2(b-a)\varepsilon_n \rightarrow \int_a^b f_n(x)\,dx + \varepsilon_n(b-a)$

$$\left| \underline{\int_a^b} f(x)\,dx - \overline{\int_a^b} f(x)\,dx \right| \le \underbrace{2(b-a)\varepsilon_n}_{\text{as } n\to\infty \;\downarrow\; 0}.$$

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$$

$$\int_a^b f_n(x)\,dx - \underbrace{(b-a)\varepsilon_n}_{n\to\infty \;\downarrow\; 0} \le \int_a^b f(x)\,dx \le \int_a^b f_n(x)\,dx + \underbrace{(b-a)\varepsilon_n}_{n\to\infty \;\downarrow\; 0}$$

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

<u>Corollary</u> Let $f_n : [a,b] \to \mathbb{R}$ $\forall n \in \mathbb{N}$
$f_n \in \mathcal{R}$.

If $f(x) = \sum_{n=0}^{\infty} f_n(x)$ converging uniformly

then $\displaystyle\int_a^b f(x)\,dx = \sum_{n=0}^{\infty} \int_a^b f_n(x)\,dx.$

$$\| $$

$$\int_a^b \left(\sum_{n=0}^{\infty} f_n(x) \right) dx$$

Recall
$$f_n(x) = \frac{\sin nx}{n} \to f(x) \equiv 0 \quad \text{uniformly on } [0, 2\pi].$$

but $f_n' = \cos nx \not\to f'(x) = 0$.

In General, $\lim_{n\to\infty} \left(\frac{d}{dx} f_n(x) \right) \neq \frac{d}{dx} \left(\lim_{n\to\infty} f_n(x) \right)$

Crucial pt: Need f_n' to converge uniformly.

Thm 7.17 Let $\forall n$ $f_n : [a,b] \to \mathbb{R}$ be diff'ble, ①

$\{f_n(x_0)\}$ converge in $\mathbb{R}$, for some $x_0 \in [a,b]$. ②

and $f_n'(x)$ converge uniformly to a function. ③

Then: $f_n(x) \to f(x)$ uniformly where
$f : [a,b] \to \mathbb{R}$
f differentiable and

$$\lim_{n\to\infty} f_n'(x) = f'(x).$$

We will prove the case where $f_n'(x)$ continuous ①'

<u>Proof</u> $f_n'(x)$ continuous th. 1'

Let $g_n(x) = f_n'(x)$; continuous th.

4 $g_n(x)$ converges to a function $g(x)$ by 3
uniformly

$g(x)$ is continuous (Thm 7.12)

5 Define $f(x) = \int_{x_0}^{x} g(t)\,dt + A$ where

$(g \text{ cont} \Rightarrow g \in \mathcal{R})$ $A = \lim_{n\to\infty} f_n(x_0)$ by 2

6 $f(x_0) = A$

$f'(x) = g(x)$ by 5 Fund. Thm of Calculus,

$$f_n'(x) = g_n(x) \longrightarrow g(x) = f'(x) \text{ uniformly}$$

<u>Fix x</u> $f_n(x) = \int_{x_0}^{x} f_n'(t)\,dt + f_n(x_0)$ (FTC)

$$\lim f_n(x) = \lim_{n\to\infty} \left(\int_{x_0}^{x} f_n'(t)\,dt + f_n(x_0) \right)$$

By Thm 7.16 + 4
$$= \int_{x_0}^{x} \lim_{n\to\infty} f_n'(t)\,dt + \lim_{n\to\infty} f_n(x_0)$$

$$= \int_{x_0}^{x} g(t)\,dt + A = f(x)$$

Hence $f_n(x) \to f(x)$ pointwise.

We want $f_n(x) \to f(x)$ uniformly.

Let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ s.t.

1) $\forall n \geq N \quad |f_n(x_0) - f(x_0)| < \frac{\varepsilon}{2}$. (by 5 + 6)

and

2) $\sup\limits_{x \in [a,b]} |g_n(x) - g(x)| \leq \frac{\varepsilon}{2|b-a|}$ by 4 $g_n \to g$ uniformly.

$\forall n \geq N$

$\forall n \geq N \ \forall x \in [a,b]$

$$|f_n(x) - f(x)| = \left| \int_{x_0}^{x} (f_n'(t) - f'(t))\,dt + (f_n(x_0) - f(x_0)) \right|$$

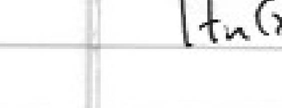
$$\leq \left| \int_{x_0}^{x} (f_n'(t) - f'(t))\,dt \right| + |f_n(x_0) - f(x_0)|$$

$\int |f| \geq \left| \int f \right|$

$$< \left| \int_{x_0}^{x} |\underbrace{f_n'(t)}_{g_n(t)} - \underbrace{f'(t)}_{g(t)}|\,dt \right| + \frac{\varepsilon}{2}$$

$$\leq \left| \int_{x_0}^{x} \frac{\varepsilon}{2} \frac{1}{b-a} \right| + \frac{\varepsilon}{2}$$

$$= \frac{\varepsilon}{2} \underbrace{\frac{|x - x_0|}{b-a}}_{\leq 1} + \frac{\varepsilon}{2} \leq \varepsilon.$$



Summarize: $f_n \to f$ uniformly

$g_n = f_n' \to f' = g$ uniformly

f_n', f' are continuous

POWER SERIES

Ex $\displaystyle\sum_{n=0}^{\infty} a_n (x-a)^n$ Let $L_0 = \limsup_{n\to\infty} \sqrt[n]{|a_n|} < \infty$

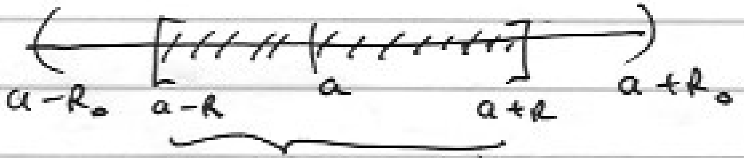
$$R_0 = \frac{1}{L_0} > 0$$

By Thms 3.33, 3.39

$\displaystyle\sum_{n=0}^{\infty} a_n (x-a)^n$ converges pointwise to a function $f(x)$ on $I = (a-R_0, a+R_0)$.

Is f continuous, differentiable, integrable?

Choose $0 < R < R_0$



$I_R = [a-R, a+R]$.

$$\sum_{n=0}^{\infty} |a_n||R|^n = \sum_{n=0}^{\infty} |a_n||a+R-a|^n$$ converges by Root test since $0 < R < R_0$. (Thm 3.39)

$\forall x \in I_R = [a-R, a+R]$ $|x-a|^n \leq R^n$

By Weierstrass M-test (Thm 7.10)

$\displaystyle\sum_{n=0}^{\infty} a_n (x-a)^n$ converges uniformly on I_R.

(Thm 7.12) $\Rightarrow$ $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ is continuous on I_R,

$\forall R$, $0 < R < R_0$.

$\Rightarrow$ $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ is continuous on J.

Caution $\sum_{n=0}^{k} a_n (x-a)^n$ may not converge to f uniformly on J.

Thm 7.16 $\Rightarrow$ $\forall\, [c,d] \subseteq J$

$$\int_c^d f(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-a)^{n+1} \Big|_{x=c}^{x=d}$$

$\forall\, x \in [c,d]$ $$\int_c^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (t-a)^{n+1} \Big|_{t=c}^{t=x}.$$

Is $f'(x) = \sum_{n=1}^{\infty} a_n \cdot n\, (x-a)^{n-1}$?

Observe that $\limsup_{n\to\infty} \sqrt[n-1]{n|a_n|} = \limsup \sqrt[n]{|a_n|} = L_0$.

$\sum_{n=1}^{\infty} a_n n (x-a)^{n-1}$ converges uniformly on each interval I_R, $\forall R$ $0 < R < R_0$.

Thm 7.17 $\Rightarrow$ $\sum_{n=1}^{\infty} a_n \cdot n \cdot (x-a)^{n-1} = f'(x)$

on I_R $\forall\, 0 < R < R_0$.

$\Rightarrow$ same is true on J

One proceeds inductively to see

$\forall k \in \mathbb{N}$ $f^{(k)}(x) = \sum_{n=k}^{\infty} a_n\, n \cdot (n-1)(n-2) \cdots (n-k+1)\, (x-a)^{n-k}$

on J.

$f \in C^{\infty}$, meaning f is infinitely many times differentiable, on J

Called f is smooth. on J.