<u>Ex</u> Uniform convergence "a compacta"

$f_n(x) = (2 - \frac{1}{n})x \longrightarrow f(x) = 2x$ pointwise on $\mathbb{R}$

$\forall L > 0$ on $[-L, L]$, f_n converges uniformly to f

$\forall \varepsilon > 0 \ \exists N = \frac{L}{\varepsilon} \ \forall n \geq N$

$\forall x \in [-L, L] \quad |f_n(x) - f(x)| = |(2 - \frac{1}{n})x - 2x|$

$= \frac{|x|}{n} \leq \frac{L}{N} = \varepsilon$

$\Rightarrow f(x) = 2x$ is continuous on $[-L, L]$. (by Thm 7.12)

Larger L' we take $L' > L$,
$f(x) = 2x$ on $[-L', L']$ is a continuous extension of $f(x) = 2x$ on $[-L, L]$.
$\Rightarrow \exists$ continuous extension $f(x) = 2x$ on $\mathbb{R}$
$f_n(x) \to f(x)$ pointwise on $\mathbb{R}$.

$f_n(x) \to f(x)$ is <u>not</u> uniform on $\mathbb{R}$

Want not$(\forall \varepsilon > 0 \exists N \ \forall n \geq N \ \forall x \in \mathbb{R} \ |f_n(x) - f(x)| \leq \varepsilon)$

$\exists \varepsilon = 1 \ \forall N \ \exists n = N, \exists x_n = 2n$

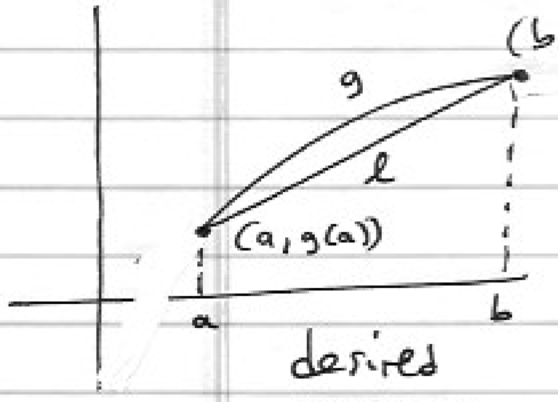
$|f_n(x_n) - f(x_n)| = \frac{|x_n|}{n} = 2 > 1 = \varepsilon$

Extended Example

Prop $g:[a,b]\to\mathbb{R}$ be twice diff'ble.

$$g''(x)\le 0 \text{ on } [a,b] \Longrightarrow g(x)\ge \underbrace{g(a)+\frac{g(b)-g(a)}{b-a}(x-a)}_{\text{line thru } (a,g(a)),\ (b,g(b))}$$

on $[a,b]$.



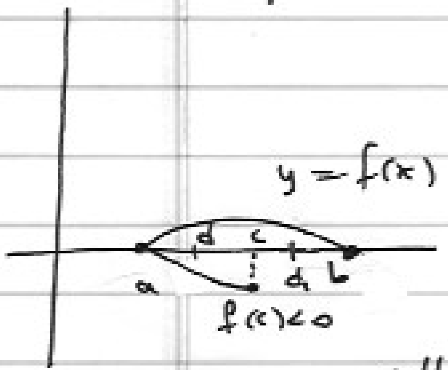
desired picture.

Proof Let

$$f(x)=g(x)-\frac{g(b)-g(a)}{b-a}(x-a)-g(a)$$

$$f(a)=f(b)=0$$

$$f''(x)=g''(x)\le 0.$$



Suppose $\exists\, c\in(a,b)$ s.t. $f(c)<0$

$$\overset{MVT}{\Longrightarrow} \exists\, d\in(a,c) \text{ s.t. } f'(d)=\frac{f(c)-f(a)}{c-a}<0$$

$$\underbrace{f''(x)\le 0 \text{ on } [a,b]}_{f' \text{ is decreasing}},\ f'(d)<0 \Longrightarrow f'(x)<0 \text{ on } [d,b].$$

But $\exists\, d_1\in(c,b)$ s.t. $f'(d_1)=\dfrac{f(c)-f(b)}{c-b}>0$

Contradiction. Hence $f(x)\ge 0$ on $[a,b]$.

#

Obs if $g''(x)<0$ on $[a,b]$ then $f(x)>0$ on (a,b). Why?

$(f(a)=0=f(b))$

One can write $g(x) \geq g(a) + \frac{g(b)-g(a)}{b-a}(x-a)$

in a slightly different way:

Recall $(1-\lambda)a + \lambda b$ parametrizes line segment $[a,b]$.

For (*) $0 \leq \lambda \leq 1$

$$g((1-\lambda)a+\lambda b) \geq g(a) + \frac{g(b)-g(a)}{b-a}\underbrace{((1-\lambda)a+\lambda b-a)}_{-\lambda a+\lambda b=\lambda(b-a)}$$

$$= g(a) + \lambda(g(b)-g(a))$$

$$= (1-\lambda)g(a) + \lambda g(b)$$

<u>Prop</u> Let $\frac{1}{p}+\frac{1}{q}=1$ for $p,q \in \mathbb{R}^+$.

Then $\forall u, v \geq 0$ $\quad uv \leq \frac{u^p}{p} + \frac{v^q}{q}$.

<u>Proof</u> If $u=0$ or $v=0$, nothing to prove.

Take $g(x) = \ln x$, $g' = \frac{1}{x}$, $g''(x) = \frac{-1}{x^2} < 0$.

By (*) $\ln\Big(\underbrace{\tfrac{1}{p}}_{1-\lambda}u^p + \underbrace{\tfrac{1}{q}}_{\lambda}v^q\Big) \geq \frac{1}{p}\ln(u^p) + \frac{1}{q}\ln v^q = \ln uv$

$g = \ln \uparrow$ strictly since $g' > 0$.

$\frac{1}{p}u^p + \frac{1}{q}v^q \geq uv$. #

Equality holds iff $u^p = v^q$.

Def^n Let $\mathscr{C}[a,b] = \{ f : [a,b] \to \mathbb{R} \mid f \text{ continuous.} \}$
& bounded.

Define: $\langle f, g \rangle_2 = \int_a^b f(x) g(x)\, dx.$

Prop $\forall f, g \in \mathscr{C}[a,b]$, $\forall p, q \in \mathbb{R}^+$, $\frac{1}{p} + \frac{1}{q} = 1$

$$\left| \int_a^b fg\, dx \right| \le \left(\int_a^b |f|^p dx \right)^{1/p} \left(\int_a^b |g|^q \right)^{1/q}$$

→ (Hölder Inequality) ($p = q = 2$ Schwarz inequality)

Proof Let $A = \left(\int_a^b |f|^p dx \right)^{1/p}$, $B = \left(\int_a^b |g|^q dx \right)^{1/q}$.

If $A = 0$, f continuous $\Rightarrow f \equiv 0$
OR
If $B = 0$, g continuous $\Rightarrow g \equiv 0$
} $fg \equiv 0$. Nothing to prove.

WLOG $AB \neq 0$.

Let $u = \frac{|f|}{A}$, $v = \frac{|g|}{B}$

$$\frac{|f||g|}{AB} = uv \le \frac{u^p}{p} + \frac{v^q}{q} = \frac{1}{p} \frac{|f|^p}{A^p} + \frac{1}{q} \frac{|g|^q}{B^q}$$

$$\int_a^b \frac{|f||g|}{AB} dx \le \int_a^b \left(\frac{1}{p} \frac{|f|^p}{A^p} \right) dx + \int_a^b \left(\frac{1}{q} \frac{|g|^q}{B^q} \right) dx$$

$$= \frac{1}{p} \frac{1}{A^p} \underbrace{\int_a^b |f|^p dx}_{A^p} + \frac{1}{q} \frac{1}{B^q} \underbrace{\int_a^b |g|^q dx}_{B^q} = 1$$

$$\int_a^b \frac{|f||g|}{AB}\,dx \le 1$$

$$\left|\int_a^b fg\,dx\right| \le \int_a^b |f||g|\,dx \le AB. = \left(\int_a^b |f|^p\right)^{1/p}\left(\int_a^b |g|^q\right)^{1/q}.$$

#.

<u>Defn</u>

- $\langle f, g\rangle_2 = \int_a^b f(x)g(x)\,dx$
- $\|f\|_2^2 = \langle f, f\rangle_2$

(Know $|\langle f, g\rangle| \le \|f\|_2\|g\|_2$ Schwarz Ineq.)

- $d_2(f,g) = \|f-g\|_2$

<u>Prop</u> $(\mathscr{C}[a,b], d_2)$ is a metric space.

<u>Proof</u> $\|f+g\|_2^2 = \int_a^b (f+g)(f+g)\,dx$

$$= \int_a^b f^2\,dx + 2\int_a^b fg\,dx + \int_a^b g^2\,dx$$

$$= \|f\|_2^2 + 2\langle f, g\rangle_2 + \|g\|_2^2$$

$$\le \|f\|_2^2 + 2\|f\|_2\|g\|_2 + \|g\|_2^2 = (\|f\|_2 + \|g\|_2)^2$$

$$\|f+g\|_2 \le \|f\|_2 + \|g\|_2.$$

#

$$\Rightarrow d_2(f,h) = \|f-h\|_2 \le \|f-g\|_2 + \|g-h\|_2 \le d_2(f,g) + d_2(g,h)$$

<u>Obs.</u>

$\forall f \in \mathscr{C}[a,b] \qquad \|f\| = \max_{x\in[a,b]} |f(x)| = A < \infty$

$$\|f\|_2^2 = \int_a^b f(x)\cdot f(x)\,dx$$
$$\leq \int_a^b A^2\,dx = (b-a)A^2.$$

$$\|f\|_2^2 \leq (b-a)\|f\|^2$$

$$\|f\|_2 \leq \sqrt{b-a}\,\|f\|.$$

Thm 7.15 $(\mathscr{C}[a,b], \|\cdot\|)$ is a complete metric space

<u>Prop</u> $(\mathscr{C}[a,b], \|\cdot\|_2)$ is <u>NOT</u> complete.

<u>Obs</u> $\{f_n\}$ Cauchy wrt $\|\cdot\|$

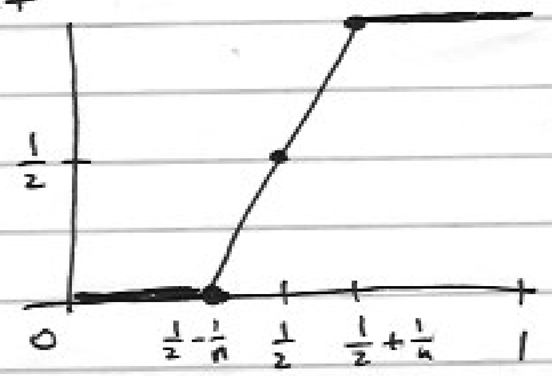
$\Rightarrow \{f_n\}$ Cauchy wrt $\|\ \|_2$

since $\underbrace{\frac{1}{\sqrt{b-a}}}_{\text{fixed \#.}} \|f_n - f_m\|_2 \leq \|f_n - f_m\| \leq \varepsilon$

We will show converse is false.

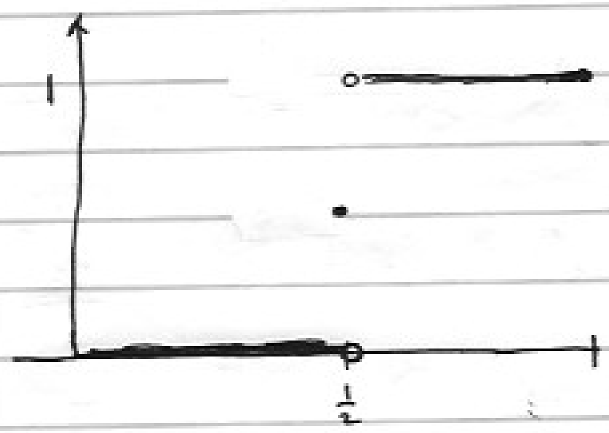
Take $[a,b]=[0,1]$.

Let



$$f_n(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{2}-\frac{1}{n} \\ \frac{n}{2}\left(x-\frac{1}{2}+\frac{1}{n}\right) & \text{linear} \\ 1 & \text{if } \frac{1}{2}+\frac{1}{n} \le x \le 1 \end{cases}$$

pointwise $\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 0 & \text{if } 0 \le x < \frac{1}{2} \\ \frac{1}{2} & \text{if } x=\frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < x \le 1 \end{cases}$

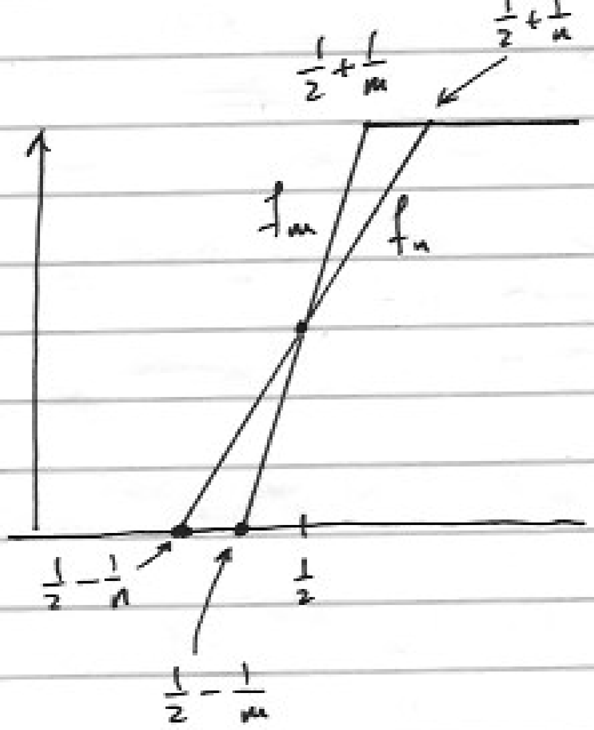


$f_n \in \mathscr{C}[a,b]$

$f \notin \mathscr{C}[a,b]$

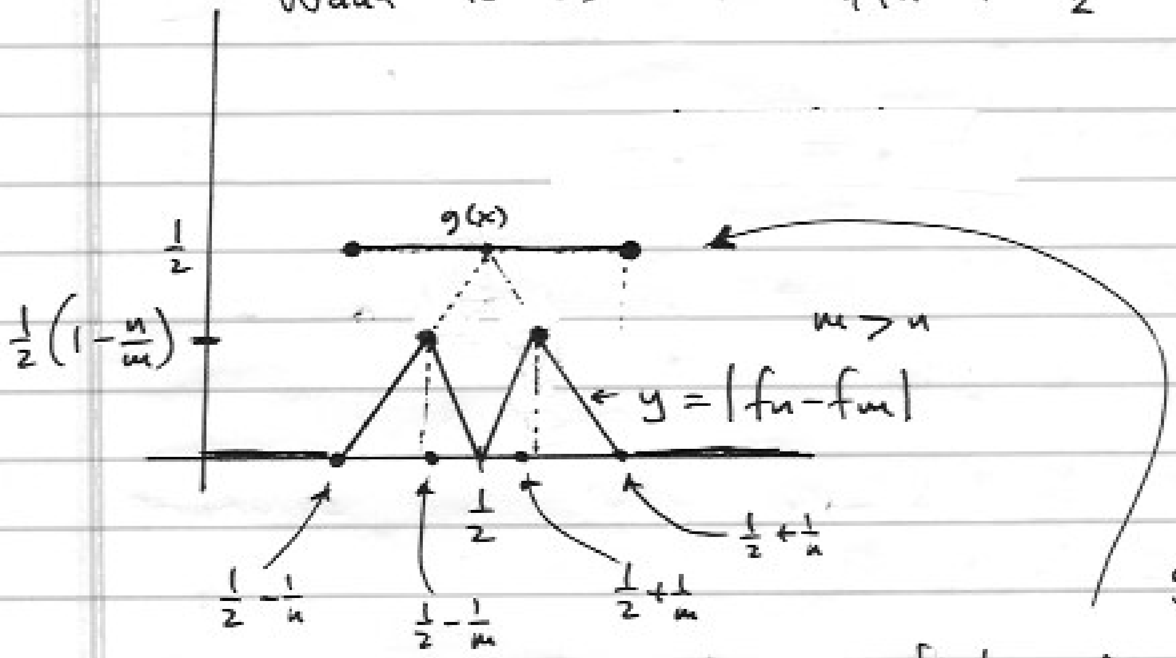
f_n can't converge to f uniformly

$\{f_n\}$ not Cauchy wrt $\|\cdot\|$.
(Thm 7.12, 7.15)



for $m>n$

Want to estimate $\|f_n - f_m\|_2$.



g(x)

$$\forall\, m \geq n \qquad |f_n - f_m| \leq \begin{cases} \frac{1}{2} & \text{if } \frac{1}{2}-\frac{1}{n} \leq x \leq \frac{1}{2}+\frac{1}{n} \\ 0 & \text{outside} \end{cases}$$

$$\|f_n - f_m\|_2^2 = \int_0^1 |f_n - f_m|^2 \leq \int_0^1 (g(x))^2\,dx = \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}+\frac{1}{n}} \frac{1}{4}\,dx$$

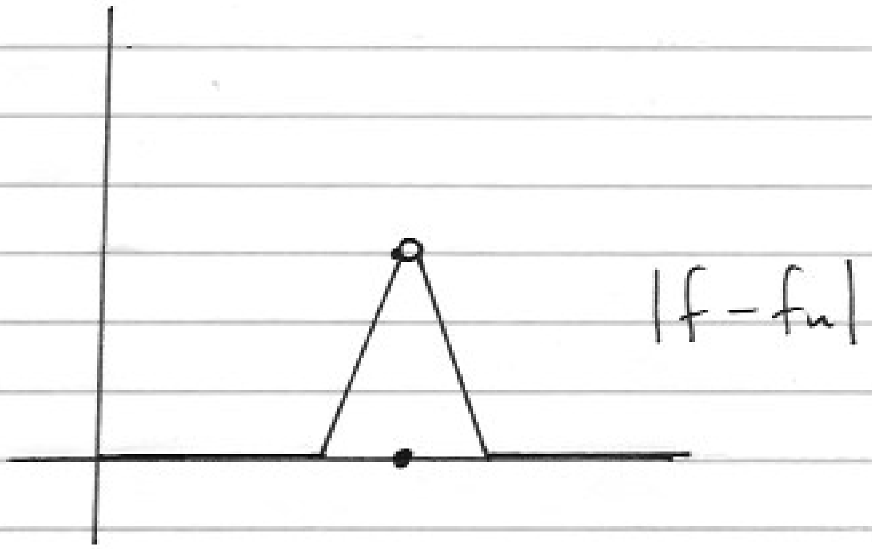
$$= \frac{2}{n}\cdot\frac{1}{4} = \frac{1}{2n}.$$

$$\|f_n - f_m\|_2 = \left(\int_0^1 |f_n - f_m|^2\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{2n}}. \quad \forall m > n.$$

$\{f_n\}$ is Cauchy wrt d_2 since

$\forall \varepsilon > 0 \ \exists\, N = \frac{1}{2\varepsilon^2} \ \ \forall n, m \geq N, \ m \geq n$

$$\|f_n - f_m\|_2 \leq \frac{1}{\sqrt{2n}} \leq \frac{1}{\sqrt{2N}} = \varepsilon$$



$$\underbrace{\int_0^1 |f-f_n|^2 dx}_{\text{R-integrable}} \le \int_0^1 (g(x))^2 dx = \frac{1}{2n}.$$

Even though $f \notin \mathscr{C}[0,1]$, one can extend $\mathscr{C}[0,1]$ to a larger space and extend d_2 to that space, to obtain a complete metric space $\mathscr{L}_2([0,1])$ (a Hilbert space). In this space one can see that $f_n \to f$ wrt d_2

But that doesnot make $\mathscr{C}[a,b]$ complete.

Compare: $(\mathbb{Q}, |\cdot|)$ extends to $(\mathbb{R}, |\cdot|)$.