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Chap VII Continue ①

***** <u>THM 7.11</u> Let $\forall n \in \mathbb{N}$ $f_n : E \to \mathbb{R}$, $E \subseteq (X, d)$,
$f : E \to \mathbb{R}$,
and $f_n \to f$ <u>uniformly</u>.

Let $x \in E'$. If $\lim_{t \to x} f_n(t) = A_n \in \mathbb{R}$, then

(i) $\lim_{n \to \infty} A_n$ exists, and

(ii) $\lim_{n \to \infty} A_n = \lim_{t \to x} f(t)$. In other words,

$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t)$$

***** <u>Corollary 7.12</u>

If $f_n : E \to \mathbb{R}$ $\forall n$; f_n continuous on E $\forall n$
$f : E \to \mathbb{R}$
and $f_n \to f$ uniformly.

Then f is continuous on E.

<u>Recall</u> $f_n(t) = t^n \to f = \underbrace{\begin{cases} 0 & \text{if } 0 < t < 1 \\ 1 & \text{if } t = 1 \end{cases}}$

converging not uniformly. not continuous

***** Proof of 7.11 (x is fixed)

(i) $\lim_{t \to x} f_n(t) = A_n \in \mathbb{R}$ $\forall n \in \mathbb{N}$ (given)

$f_n \to f$ uniformly (given)

$\Rightarrow$ $\{f_n\}$ is uniformly Cauchy (Thm 7.8)

$\forall \varepsilon > 0$ $\exists N$ $\forall n, m \geq N$ $\forall t \in E$ $|f_n(t) - f_m(t)| \leq \varepsilon$

$$-\varepsilon \leq f_n(t) - f_m(t) \leq \varepsilon$$

$$\lim_{t \to x} \downarrow \qquad \downarrow \lim_{t \to x}$$

$$-\varepsilon \leq A_n - A_m \leq \varepsilon$$

$\forall \varepsilon > 0$ $\exists N$ $\forall n, m \geq N$ $|A_n - A_m| \leq \varepsilon$.

$\{A_n\}$ is Cauchy sequence of real numbers

$\mathbb{R}$ Complete $\Rightarrow$ $\exists \lim_{n \to \infty} A_n = A \in \mathbb{R}$ for some A

(ii) $\lim_{n \to \infty} A_n = \underset{\text{Did}}{A} \overset{?}{=} \underset{\text{want}}{\lim_{t \to x} f(t)}$

$\forall t$ $\forall n$

$$|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|$$

Let $\varepsilon > 0$ be given. Choose $n \in \mathbb{N}$ s.t.

① $|f(t) - f_n(t)| \le \varepsilon/3 \quad \forall t \in E$

and $\qquad (f_n \to f \text{ unif.})$

② $|A_n - A| \le \frac{\varepsilon}{3} \qquad (\lim_{n\to\infty} A_n = A)$

For this n, $\lim_{t \to x} f_n(t) = A_n$ (hypothesis)

($\varepsilon > 0$ was given) $\exists\, V$ open $\subseteq (X, d)$, $x \in V$ s.t

$\forall t \in V \cap E - \{x\}$, $\quad |f_n(t) - A_n| \le \frac{\varepsilon}{3}$ def. of limit

Summarizing: $\forall \varepsilon > 0\ \exists\, V$ open $\subseteq (X, d)$, $x \in V$
$\forall t \in V \cap E - \{x\}$

$$|f(t) - A| \le |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|$$

$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

$$\lim_{t \to x} f(t) = A.$$

$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{t \to x} f(t) = A = \lim_{n \to \infty} A_n$$

$$= \lim_{n \to \infty} \lim_{t \to x} f_n(t) \quad \#$$

Proof of Corollary 7.12

Let $f_n \to f$ uniformly, f_n continuous $\Big\}$ $f_n, f : E \to \mathbb{R}$

(want f is continuous)

$$\lim_{t\to x} f(t) = \lim_{t\to x} \underbrace{\lim_{n\to\infty} f_n(t)}_{f_n \to f \text{ uniformly}}$$

$$\underset{\text{Thm 7.11}}{=} \lim_{n\to\infty} \lim_{t\to x} f_n(t) \underset{f_n \text{ continuous}}{=} \lim_{n\to\infty} f_n(x) \underset{f_n \to f}{=} f(x)$$

<u>**Defn**</u> Let (X, d) be a metric space.

$$\mathscr{C}(X) = \left\{ f : X \to \mathbb{R} \;\middle|\; \begin{array}{l} f \text{ continuous on } X \text{ and} \\ f \text{ is bounded} \end{array} \right\}$$

$\forall f \in \mathscr{C}(X)$, define $\|f\| = \sup_{x\in X} |f(x)| < \infty$,

and $d_{\mathscr{C}}(f, g) = \|f - g\|$.

Prop $(\mathscr{C}(X), d_{\mathscr{C}})$ is a metric space

Proof: (i) $\|f\| \geq 0$

$\|f\| = 0 \iff \sup_{x\in X} |f(x)| = 0 \iff f(x) \equiv 0$ on X.

Hence $d_{\mathscr{C}}(f,g) = \|f-g\| \geq 0$ and
$d_{\mathscr{C}}(f,g) = 0 \iff \|f-g\| = 0 \iff f-g \equiv 0$ on X
$\iff f \equiv g$ on X.

(ii) $d_{\mathscr{C}}(f,g) = \|f-g\| = \|g-f\| = d_{\mathscr{C}}(g,f)$.

(iii) $\forall f,g \in \mathscr{C}(X)$ $\forall x \in X$ $|f(x)+g(x)| \leq |f(x)|+|g(x)|$

$\forall x \in X$ $|f(x)| \leq \|f\| < \infty$, and
$|g(x)| \leq \|g\| < \infty$

$\forall x \in X$ $|f(x)+g(x)| \leq \|f\| + \|g\| < \infty$
$\sup_{x\in X} |f(x)+g(x)| = \|f+g\| \leq \|f\|+\|g\|$

Consequently $\forall f,g,h \in \mathscr{C}(X)$

$d_{\mathscr{C}}(f,h) = \|f-h\| = \|f-g+g-h\|$
$\leq \|f-g\| + \|g-h\| = d_{\mathscr{C}}(f,g) + d_{\mathscr{C}}(g,h)$.

$d_{\mathscr{C}}(f,h) \leq d_{\mathscr{C}}(f,g) + d_{\mathscr{C}}(g,h)$.

***** <u>Theorem 7.15</u> Let (X, d) be a metric space. Then $(\mathcal{C}(X), d_{\mathcal{C}})$ is a complete metric space.

<u>Proof</u> Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{C}(X)$

$\forall n$, $f_n : X \longrightarrow \mathbb{R}$ continuous and bounded.

$\forall \varepsilon > 0 \; \exists N \; \forall n, m \geq N$

$$\varepsilon \geq d_{\mathcal{C}}(f_n, f_m) = \|f_n - f_m\| = \sup_{x \in X} |f_n(x) - f_m(x)|$$

$$\forall x \in X \quad |f_n(x) - f_m(x)| \leq \varepsilon$$

Hence $\{f_n(x)\}$ satisfies CCUC

Thm 7.8 $\Longrightarrow \exists f : X \longrightarrow \mathbb{R}$, $f_n \longrightarrow f$ uniformly.

f_n continuous on E $\underset{\text{Thm 7.12}}{\Longrightarrow}$ f is continuous on E

Need to show $f \in \mathcal{C}(X)$. (Need bounded)

$\forall \varepsilon > 0 \; \exists N \; \forall n \geq N \; \forall x \in X$

$$|f_n(x) - f(x)| \leq \varepsilon. \text{ (Since } f_n \longrightarrow f \text{ unif.)}$$

$$|f(x)| - |f_n(x)| \leq |f_n(x) - f(x)| \leq \varepsilon.$$

$$\forall x \in \mathbb{X} \quad |f(x)| \leq \varepsilon + |f_n(x)| \leq \varepsilon + \|f_n\| < \infty.$$

$$\|f\| = \sup_{x \in \mathbb{X}} |f(x)| \leq \varepsilon + \|f_n\|$$

$\Rightarrow f$ is bounded, & continuous $\Big\}\ f \in \mathscr{C}(\mathbb{X})$

$\forall \varepsilon > 0 \ \exists N \ \forall n \geq N$

$$d_{\mathscr{C}}(f, f_n) = \|f - f_n\| = \sup_{x \in \mathbb{X}} |f_n(x) - f(x)| \leq \varepsilon$$

$f_n \to f$ with respect to $d_{\mathscr{C}}$.

IMPORTANT REMARK

$(\mathbb{X}, d)$ need not be complete, but

$(\mathscr{C}(\mathbb{X}), d_{\mathscr{C}})$ is complete.

Crucial pt. $f: \mathbb{X} \to \mathbb{R}$ complete.

Example

$$\sum_{n=1}^{\infty} \frac{e^x \sin nx}{n^2} \qquad 0 \le x \le 2\pi.$$

On $0 \le x \le 2\pi$, $|e^x| \le e^{2\pi} \le 3^7$

$|\sin nx| \le 1$

$$\left| \frac{e^x \sin nx}{n^2} \right| \le \frac{C}{n^2} = M_n$$

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{C}{n^2} < \infty$$

Thm 7.10. Weierstrass M-test $\Longrightarrow \sum_{n=1}^{\infty} \frac{e^x \sin nx}{n^2} = f(x)$ exists on $[0, 2\pi]$

The convergence of the series is uniform, absolutely convergent.

Thm 7.12: $f(x)$ is continuous on $[0, 2\pi]$.

IMPORTANT TRICK: $\forall L \in \mathbb{R}^+, (L < \infty)$

Same proof above works on $[-L, L]$, $M_n = \frac{e^L}{n^2}$

partial sums $\quad f_n = \underbrace{\sum_{k=1}^{n} \frac{e^x \sin kx}{k^2}}_{\text{continuous}} \longrightarrow f(x)$ uniformly on $[-L, L]$

$f(x)$ continuous on $[-L, L]$.

$\Longrightarrow \exists f(x)$ continuous on $(-\infty, \infty)$, $f_n \to f$ pointwise.

BUT f_n may not converge uniformly to f on $\mathbb{R}$ $\left(\sup_{\mathbb{R}} |e^x \sin x| = +\infty \right)$