CHAPTER VII

Defn 7.1, 7.7

Let $f_n : E \longrightarrow \mathbb{R}$ be defined for all $n \in \mathbb{N}$.

$\{f_n(x)\}_{n=1}^{\infty}$ is called a sequence of functions.

Let $f : E \longrightarrow \mathbb{R}$ be given.

- $f_n(x)$ is said to converge to $f(x)$ pointwise if $\forall x \in E$ $\forall \varepsilon > 0$ $\exists N = N(x, \varepsilon)$ s.t.
$$\forall n \geq N \quad |f_n(x) - f(x)| \leq \varepsilon$$

- $f_n(x)$ is said to converge to $f(x)$ uniformly on E if $\forall \varepsilon > 0$ $\exists N = N(\varepsilon)$ s.t.
$$\forall n \geq N, \forall x \in E \quad |f_n(x) - f(x)| \leq \varepsilon.$$

Obs $\left. \begin{array}{l} f_n \longrightarrow f \text{ uniformly} \\ \text{on } E \end{array} \right\} \Longrightarrow \begin{array}{l} f_n \to f \text{ pointwise} \\ \text{for each } x \in E \end{array}$

Ex 1. $f_n(x) = x^n : (-1, 1] \longrightarrow \mathbb{R}$, $n \in \mathbb{N}$.

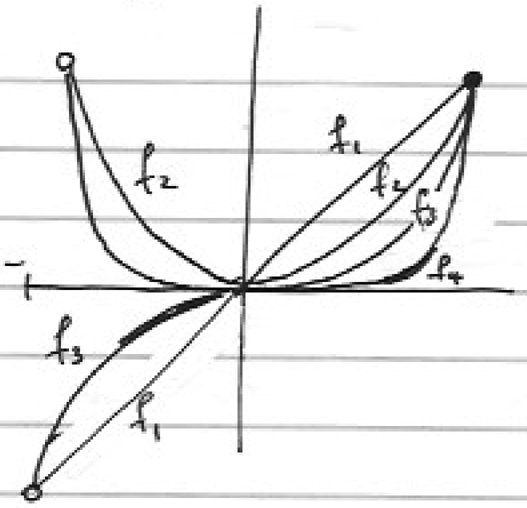
. Fix x: If $|x| < 1$, $x^n \longrightarrow 0$ as $n \to \infty$.

If $x = 1$ $1^n \longrightarrow 1$ as $n \to \infty$.

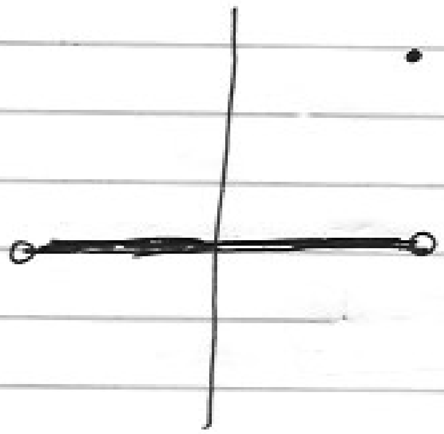
$$\text{Let } f(x) = \begin{cases} 0 & \text{if } -1 < x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

$f_n(x)$ converges to $f(x)$ pointwise.

$y = f_n(x) = x^n$



$y = f(x)$



$\forall n \in \mathbb{N}$ $f_n(x)$ is continuous on $(-1,1]$

$f(x)$ is <u>not</u> continuous at 1.

$$0 = \lim_{\substack{x \to 1^- \\ x<1 \Rightarrow}} \underbrace{\lim_{n\to\infty} f_n(x)}_{0} \neq \lim_{n\to\infty} \underbrace{\lim_{x \to 1^-} f_n(x)}_{1} = 1$$

(b) $f_n(x) \longrightarrow f(x)$ pointwise but <u>not</u> uniformly:

Want to show:

<u>WTS</u> $\begin{cases} \text{not}(\forall \varepsilon>0\ \exists N\ \forall n \geq N\ \forall x \in E \quad |f_n(x)-f(x)| \leq \varepsilon) \\ (\exists \varepsilon>0\ \forall N\ \exists n \geq N\ \exists x \in E \quad |f_n(x)-f(x)| > \varepsilon) \end{cases}$

Let $\varepsilon = \frac{1}{2}$. Recall $\sqrt[n]{1/2} \nearrow 1$

$\sqrt[n]{1/2} < 1 \quad \forall n \in \mathbb{N}.$

$\forall N\ \exists n=N \quad \exists x_n \quad \sqrt[n]{1/2} < x_n < 1$

$\frac{1}{2} < (x_n)^n < 1$

$$|f_n(x_n) - f(x_n)| = |(x_n)^n - 0| > \frac{1}{2}.$$

Ex 2

Define $S_{n,m} = \dfrac{3n+4m}{5n+m} \quad \forall n,m \in \mathbb{N}$

$$\lim_{m\to\infty} \lim_{n\to\infty} \frac{3n+4m}{5n+m} = \lim_{m\to\infty} \frac{3}{5} = \frac{3}{5}$$

$$\#$$

$$\lim_{n\to\infty} \lim_{m\to\infty} \frac{3n+4m}{5n+m} = \lim_{n\to\infty} 4 = 4$$

Ex 3 $\displaystyle\sum_{n=0}^{\infty} x(1-x)^n = \begin{cases} 0 & \text{if } x=0 \\ 1 & \text{if } 0<x<1 \end{cases}$

$\frac{1}{1-(1-x)} = \frac{1}{x}$

$\forall M \in \mathbb{N}$ $\displaystyle\sum_{n=0}^{M} x(1-x)^n$ are continuous on $[0,1)$

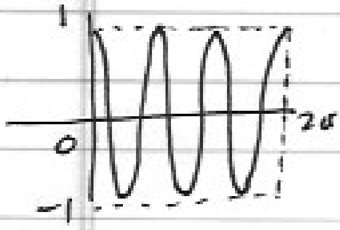
$\displaystyle\sum_{n=0}^{\infty} x(1-x)^n$ is not continuous at 0.

Ex 4 $f_n(x) = \dfrac{\sin nx}{n} \longrightarrow 0 = f(x)$ uniformly on $[0,2\pi]$

$\forall \varepsilon > 0 \ \exists N = \frac{1}{\varepsilon} \ \forall n \geq N \ \left|\frac{\sin nx}{n} - 0\right| \leq \frac{1}{n} \leq \frac{1}{N} = \varepsilon.$

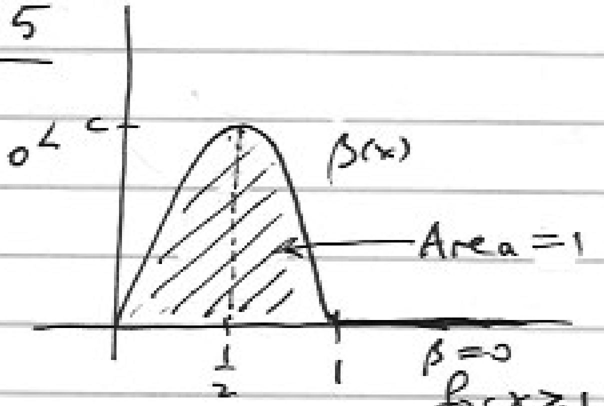
$f_n'(x) = \cos nx.$ $\quad f'(x) = 0$

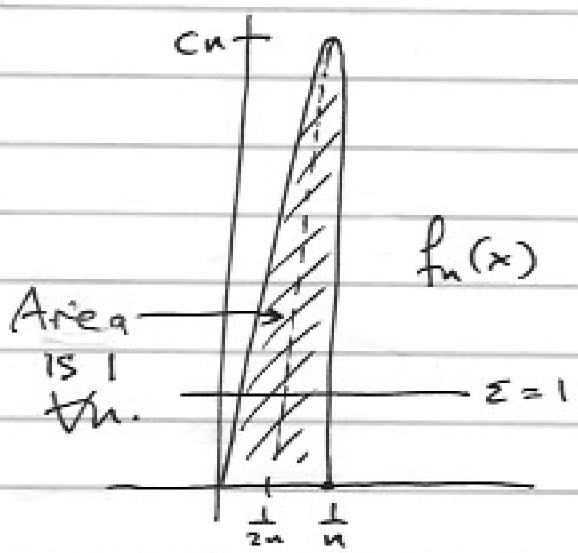
$f_n'(x) \not\to f'(x)$ for $x \neq 0, 2\pi$



not even pointwise

Ex 5





$$\left.\begin{aligned} f_n(x) &= n\beta(nx) \\ f(x) &= 0 \end{aligned}\right\}$$

- $\forall x > 0 \; \exists N \in \mathbb{N}, \; 0 < \frac{1}{N} < x, \; f_n(x) = 0 \; \forall n \geq N$

  $\lim\limits_{n\to\infty} f_n(x) = 0$, also true for $x = 0$ since $f_n(0) = 0 \; \forall n$.

  $\lim\limits_{n\to\infty} f_n(x) = f(x)$ pointwise (for each fixed x)

- Not uniform convergence:

  $\exists \varepsilon = 1, \forall N \exists n$ large s.t. $\max f_n(x) = cn > 1$

  $$\Rightarrow \left| f_n\left(\tfrac{1}{2n}\right) - \underbrace{f\left(\tfrac{1}{2n}\right)}_{0} \right| = \underset{\substack{\vee \\ \varepsilon = 1}}{cn}$$

- $$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} 1 = 1$$

  $$\nparallel$$

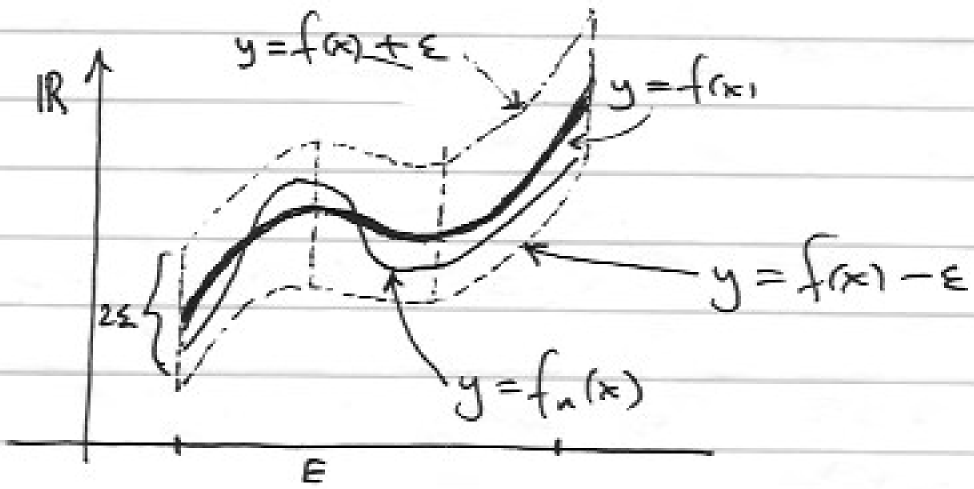
  $$\int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 0\,dx = 0$$

Obs

Geometrically

$\forall x \in E \quad |f_n(x) - f(x)| < \varepsilon$ means

$\forall x \in E: f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon$



Graph of $f_n(x)$ lies inside a ribbon of height 2ε centered along the graph of $y = f(x)$

CCUC

Thm 7.8 Let $f_n : E \to \mathbb{R}$, $\forall n \in \mathbb{N}$.

(i) $\Leftrightarrow$ (ii) where

(i) f_n converges uniformly on E to a function $f : E \to \mathbb{R}$. i.e.
$\forall \varepsilon > 0\ \exists N\ \forall n \geq N\ \forall x \in E\ |f_n(x) - f(x)| \leq \varepsilon$

(ii) $\{f_n\}$ satisfy Cauchy Criterion for uniform Convergence
$\forall \varepsilon > 0\ \exists N\ \forall n, m \geq N, \forall x \in E\ |f_m(x) - f_n(x)| \leq \varepsilon$.
CCUC

Proof: (i) $\Rightarrow$ (ii)

$$\forall \varepsilon > 0\ \exists N\ \forall n \geq N\ \forall x \in E\ |f_n(x) - f(x)| \leq \frac{\varepsilon}{2}$$

If $m, n \geq N$, then $\forall x \in E$

$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| \\ &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

(ii) $\Rightarrow$ (i) Given: $\{f_n\}$ is uniformly Cauchy
We do not have f yet.

Fix $x \in E$.
$\forall \varepsilon > 0 \ \exists N \ \forall m, n \geq N \quad |f_n(x) - f_m(x)| \leq \varepsilon.$

Since x is fixed $\{f_n(x)\}$ is a Cauchy sequence of real numbers. $\mathbb{R}$ is complete.
$\Rightarrow \exists$ a limit in $\mathbb{R}$

Define $\lim\limits_{n \to \infty} f_n(\underset{\substack{\uparrow \\ x \text{ fixed}}}{x})$ to be $f(x)$.

Now $f(x)$ defined. $\lim\limits_{n\to\infty} f_n(x) = f(x)$ pointwise

CCUC: $\forall \varepsilon > 0 \overset{*}{\ } \exists N \ \forall n, m, \ m \geq n \geq N \ \forall x \in E$

independent of x.

$$|f_n(x) - f_m(x)| \leq \varepsilon$$
$$-\varepsilon \leq f_n(x) - f_m(x) \leq \varepsilon$$
$$f_m(x) - \varepsilon \leq f_n(x) \leq f_m(x) + \varepsilon$$

n fixed
$m \geq n$
$m \to \infty$

$m \to \infty$ pointwise $\qquad m \to \infty$ pointwise

$$f(x) - \varepsilon \leq f_n(x) \leq f(x) + \varepsilon$$ True for all $x \in E$

$$|f_n(x) - f(x)| \leq \varepsilon$$ given, independent of x

same as $*$ above

$$\forall \varepsilon > 0 \ \exists N = N(\varepsilon) \ \forall n \geq N \ \forall x \in E$$
$$|f_n(x) - f(x)| \leq \varepsilon.$$

That is uniform convergency.

Thm 7.9

Let $\lim_{n\to\infty} f_n(x) = f(x)$ pointwise.

Let $M_n = \sup_{x\in E} |f_n(x) - f(x)|$.

① $f_n \to f$ uniformly $\iff \lim_{n\to\infty} M_n = 0$

Proof $\forall x \in E \ |f_n(x) - f(x)| \le \varepsilon \iff 0 \le M_n \le \varepsilon$.

Write out defns of each side of ①

$\forall \varepsilon > 0 \ \exists N \ \forall n \ge N \ \forall x \in E \ |f_n(x) - f(x)| \le \varepsilon$ becomes $\forall \varepsilon > 0 \ \exists N \ \forall n \ge N, 0 \le M_n \le \varepsilon$

*** WEIERSTRASS M-TEST Thm 7.10

Let $f_n : E \to \mathbb{R}$, $\forall n$, & $M_n = \sup_{x\in E} |f_n(x)| \ge 0$

$\sum_{n=1}^{\infty} M_n < \infty \implies \sum_{n=1}^{\infty} f_n(x)$ converges uniformly and absolutely to a limit function f.

This is a consequence of Thm 7.8

Proof: $\sum_{n=1}^{\infty} M_n < \infty \implies$ Cauchy Criterion for series in $\mathbb{R}$

$\forall \varepsilon > 0 \ \exists N \ \forall m > n \ge N \ \left| \sum_{k=n+1}^{m} M_k \right| < \varepsilon$

$M_k \ge 0$

$\implies \forall x \in E \ \left| \sum_{k=n+1}^{m} f_k(x) \right| \le \sum_{k=n+1}^{m} |f_k(x)| \le \sum_{k=n+1}^{m} M_k < \varepsilon$

$\implies$ CCUC for the partial sums of series of functions holds

Thm 7.8 $\implies \sum_{n=1}^{\infty} f_n(x)$ and $\sum_{n=1}^{\infty} |f_n(x)|$ converge uniformly.