L'Hospital's Rule Thm (5.13)

Let $f, g : (a,b) \to \mathbb{R}$ be differentiable,
$g'(x) \neq 0$ on (a,b), where $-\infty \leq a < b \leq +\infty$

If

(i) $\dfrac{f'(x)}{g'(x)} \to A$ as $x \to a$, and

(ii) (either (a) $\left.\begin{matrix} f(x) \to 0 \\ g(x) \to 0 \end{matrix}\right\}$ as $x \to a$ OR b) $g(x) \to +\infty$ as $x \to a$)

Then $\dfrac{f(x)}{g(x)} \to A$.

Proof HW to read.

TAYLOR POLYNOMIALS

Defn Let $f : [a,b] \to \mathbb{R}^1$ be given, $\alpha \in [a,b]$.
Let $m \in \mathbb{N}$ s.t.
$f', f'', f''', \dots f^{(m)}$ are defined on $[a,b]$.

Then:

$$P_{m,\alpha}(t) = \sum_{k=0}^{m} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$$ is called

the m^{th} degree Taylor polynomial of f about α.

(Recall : $f^{(k)} = k^{\text{th}}$ derivative of f, e.g. $f''' = f^{(3)}$)

Ex $f(x) = e^x$

$$P_{3,0}(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6}.$$

TAYLOR'S THEOREM Let $f: [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$ be s.t. (i) $f^{(n-1)}$ is continuous on $[a,b]$ and (ii) $f^{(n)}$ exists on (a,b).
Let $\alpha \in (a,b)$, $\beta \in [a,b]$.
Then

$$f(\beta) = P_{n-1,\alpha}(\beta) + \frac{f^{(n)}(c)}{n!}(\beta - \alpha)^n$$

for some c between α and β.

Observe that

(1) $\left.\begin{array}{l} f^{(n-1)} \text{ continuous} \\ \text{on } [a,b] \end{array}\right) \overset{\text{Thm } 5.2}{\Longrightarrow}$ $f', f'', f''', \ldots f^{(n-2)}$ all exist and all are continuous on $[a,b]$.

(2) For $n = 1$, Taylor's Thm is MVT:
(mean Value Thm)

$$P_{0,\alpha} = \sum_{k=0}^{0} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k = f(\alpha)$$

Taylor's Thm $f(\beta) = f(\alpha) + f'(c)(\beta - \alpha)$

$$\frac{f(\beta) - f(\alpha)}{\beta - \alpha} = f'(c)$$ for some c between α & β.

Proof of Taylor's Thm $n \geq 2$

This is done by repeated applications of MVT
(We proved GMVT independently; GMVT $\Rightarrow$ MVT)

Fix n, α, β, $\alpha \neq \beta$.

Set $P(t) = \displaystyle\sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$ a polynomial of degree $n-1$.

$f(\beta)$, $P(\beta)$, $\alpha \neq \beta$ known; choose M s.t.

$$f(\beta) = P(\beta) + M(\beta-\alpha)^n.$$

Let $g(t) = f(t) - P(t) - M(t-\alpha)^n$.

$f', f'', \ldots f^{(n-1)}$ continuous on $[a,b]$, $f^{(n)}$ exists on (a,b)
$\Rightarrow g', g'', \ldots g^{(n-1)}$ " " " $g^{(n)}$ " " "
(since $P(t) + M(t-\alpha)^n$ is a polynomial)

So MVT is applicable to all of
$g, g', g'', g''', \ldots, g^{(n-1)}$.

By MVT for g $\exists\, \beta_1$ between α and β s.t.

$$g'(\beta_1) = \frac{g(\beta) - g(\alpha)}{\beta - \alpha}.$$

$$g(\alpha) = f(\alpha) - P(\alpha) - 0 = f(\alpha) - f(\alpha) = 0$$
$$g(\beta) = f(\beta) - P(\beta) - M(\beta-\alpha)^n = 0$$

$\Rightarrow \exists\, \beta_1$ between α & β s.t.
$g'(\beta_1) = 0$

$$g(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k - M(t-\alpha)^n$$

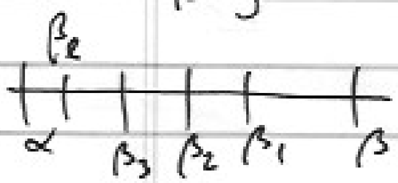
$$g'(t) = f'(t) - \sum_{k=1}^{n-1} \frac{f^{(k)}(\alpha)}{k!} k (t-\alpha)^{k-1} - Mn(t-\alpha)^{n-1}$$

$$= f'(t) - \sum_{k=1}^{n-1} \frac{f^{(k)}(\alpha)}{(k-1)!}(t-\alpha)^{k-1} - Mn(t-\alpha)^{n-1}.$$

$g'(\alpha) = f'(\alpha) - f'(\alpha) = 0$

$g'(\beta_1) = 0$ (by the first step.)

$\underset{\text{for } g'}{\text{MVT}} \Rightarrow \exists\, \beta_2$ between α and β_1 s.t. $g''(\beta_2) = 0$



For $\ell = 1, 2, 3, .. n-1$

$$g^{(\ell)}(t) = f^{(\ell)}(t) - \sum_{k=\ell}^{n-1} \frac{f^{(k)}(\alpha)}{(k-\ell)!}(t-\alpha)^{k-\ell} - Mn(n-1)\cdots(n-\ell+1)(t-\alpha)^{n-\ell}$$

$g^{(\ell)}(\alpha) = f^{(\ell)}(\alpha) - f^{(\ell)}(\alpha) = 0$

$g^{(\ell)}(\beta_\ell) = 0$ by the choice of β_ℓ in the previous step.

$\underset{\text{for } g^{(\ell)}}{\text{MVT}} \Rightarrow \exists\, \beta_{\ell+1}$ between α and β_ℓ s.t.

$$g^{(\ell+1)}(\beta_{\ell+1}) = 0$$

For $l = n$

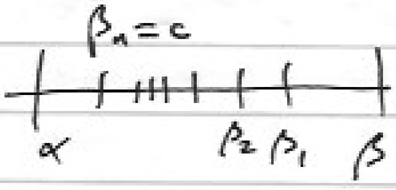
$\exists \beta_n$ s.t. $g^{(n)}(\beta_n) = 0$ for some β_n between α and β_{n-1}.

$$g^{(n)}(t) = f^{(n)}(t) - n!\,M.$$

$\left(P^{(n)} \equiv 0 \text{ since degree } P^{(n)} = n-1 \right)$

Take $c = \beta_n$

$$0 = g^{(n)}(c) = f^{(n)}(c) - n!\,M$$



$$M = \frac{f^{(n)}(c)}{n!}$$

$$f(\beta) = P(\beta) + \frac{f^{(n)}(c)}{n!}(\beta - \alpha)^n$$

Since this is how M was chosen.

c is between α and β. #

Remark: Why do we care about $M = \dfrac{f^{(n)}(c)}{n!}$?

$$\underbrace{|\, f(\beta) - P(\beta) \,|}_{\text{error of the approximation}} = \left| \frac{f^{(n)}(c)}{n!} \right| \cdot |\beta - \alpha|^n.$$

$f(\beta)$: want

$P(\beta)$: approximation

If we have a rough estimate on $|f^{(n)}(t)| \le N$ between α & β,

then $|\text{error}| \le \dfrac{N}{n!} |\beta - \alpha|^n$

Easy example

$$f(x) = e^x, \quad P_{3,0}(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6}.$$

$$e^{0.01} \approx 1 + 0.01 + \frac{(0.01)^2}{2} + \frac{(0.01)^3}{6} = \underbrace{1.0100501\overline{6}}_{\text{how good is this estimate?}}$$

$$\frac{d}{dx} e^x = e^x$$

$$1 = e^0 \le e^x \le e^1 \le 3 = N \quad \text{if} \quad 0 \le x \le 0.01$$

$$|\text{error}| \le \frac{3}{4!}(0.01)^4 = 0.00000000125$$

$$e^{0.01} \cong 1.010050167 \pm 0.000000002$$

PTO

MEAN VALUE THM for Vector Valued (one variable) functions.

Let $\vec{f} : [a,b] \longrightarrow \mathbb{R}^k$

$\vec{f}(t) = (f_1(t), f_2(t), .. f_k(t))$,

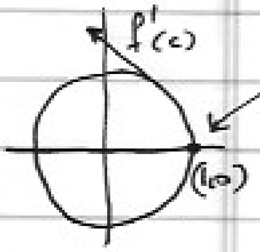
$i = 1, 2, 3, .. k$ $\quad f_i : [a,b] \longrightarrow \mathbb{R}^1$ component functions.

$\vec{f}$ diff'ble $\Longleftrightarrow \forall_i \; f_i$ is differentiable

Example MVT is false in $k > 1$.

$$\text{“} \frac{f(b) - f(a)}{b - a} = f'(c) \text{ for some } c \in (a,b) \text{”}$$

may not hold under f continuous on $[a,b]$ & f diff'ble on (a,b).



Take $f(t) = (\cos t, \sin t)$, $\quad 0 \leq t \leq 2\pi$

$f(0) = (1, 0) = f(2\pi)$

$f(2\pi) - f(0) = (0, 0)$

$f'(t) = (-\sin t, \cos t)$, $\quad \|f'(t)\| = 1$

$\forall c \in (0, 2\pi)$

$$\frac{f(2\pi) - f(0)}{2\pi - 0} = (0,0) \neq (-\sin c, \cos c).$$

Why can't we apply MVT to each component?

$\exists c_1 \in (0, 2\pi)$, $\big(\underbrace{f_1(2\pi) - f_1(0)}_{0}\big)\frac{1}{2\pi} = f_1'(c_1) = -\sin\pi = 0$

$\exists c_2 \in (0, 2\pi)$, $\big(f_2(2\pi) - f_2(0)\big)\frac{1}{2\pi} = f_2'(c_2) = \cos\frac{\pi}{2} = 0$

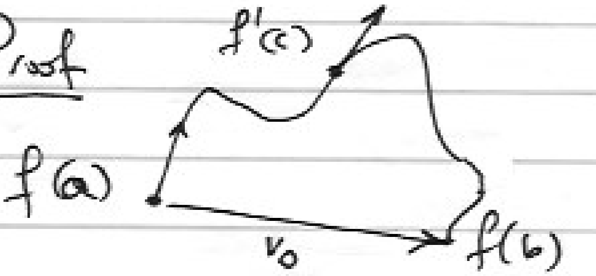
but $\pi = c_1 \neq c_2 = \frac{\pi}{2}$.

Mean Value Thm in higher dimensions

For $b > a$, Let $\vec{f} : [a,b] \to \mathbb{R}^k$ be continuous on $[a,b]$
& be differentiable on (a,b).

Then $\exists\, c \in (a,b)$ s.t.
$$\|f(b) - f(a)\| \le (b-a)\,\|f'(c)\|.$$

Proof



Let $v_0 = f(b) - f(a)$

Set $\varphi(t) = v_0 \cdot f(t)$ ← dot product.

$\varphi : [a,b] \to \mathbb{R}$

φ continuous on $[a,b]$, diff'ble on (a,b).
$$\varphi'(t) = v_0 \cdot f'(t)$$

MVT $k=1 \Rightarrow \exists\, c \in (a,b)$ s.t. $\varphi(b) - \varphi(a) = (b-a)\,\varphi'(c)$

$$\|\varphi(b) - \varphi(a)\| = \|v_0 \cdot f(b) - v_0 \cdot f(a)\| = \|v_0 \cdot (\underbrace{f(b) - f(a)}_{v_0}\| = \|v_0\|^2.$$

$$\|\varphi'(c)\,(\overset{\in \mathbb{R}}{b-a})\| = |b-a|\,\|v_0 \cdot f'(c)\| \le |b-a|\,\|v_0\|\,\|f'(c)\|$$

Cauchy-Schwartz

$$\|v_0\|^2 = \|\varphi(b) - \varphi(a)\| = \|(b-a)\,\varphi'(c)\| \le |b-a|\,\|v_0\|\,\|f'(c)\|.$$

If $\|v_0\| \ne 0$, $\|f(b) - f(a)\| = \|v_0\| \le |b-a|\,\|f'(c)\|$.

If $\|v_0\| = 0$, $\|f(b) - f(a)\| = 0 \le |b-a|\,\|f'(c)\|$ $\forall c$.