Chap V

Differentiation Continue

Defn Let $f:(X, d_X) \to \mathbb{R}$. f is said to have a local maximum at $p \in X$ if $\exists \delta > 0$ s.t. $\forall x \in N_\delta(p)$ $f(x) \le f(p)$.

Similarly one defines local minimum.

Theorem: 1st Derivative Test.

Let $f:[a,b] \to \mathbb{R}$. If f has a local maximum (or local minimum) at p where

(i) $p \in (a,b)$ (interior pt), and

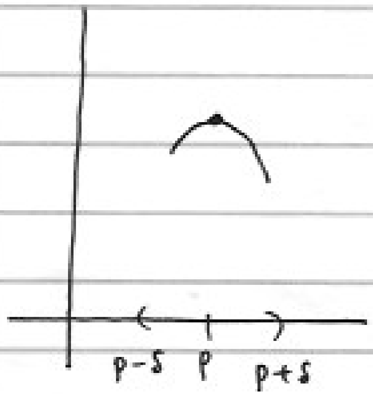
(ii) f is differentiable at p,

then $f'(p) = 0$.

Proof

(we only do local max case; local min case HW)

Choose $\delta > 0$ s.t. $\forall x \in N_\delta(p)$, $f(x) \le f(p)$, and make sure $(p-\delta, p+\delta) \subseteq (a,b)$, by choosing δ smaller if necessary.



If $p-\delta < t < p$: $\dfrac{f(t)-f(p)}{t-p} \ge 0$

$$f'(p) = \lim_{t \to p} \frac{f(t)-f(p)}{t-p} \text{ exists.}$$

$$= \lim_{t \to p^-} \frac{f(t)-f(p)}{t-p} \ge 0 \quad ①$$

By Lemma (April 15) p 7

have: ($f(t) \le f(p)$)

If $p < t < p+\delta$, then $\dfrac{f(t)-f(p)}{t-p} \le 0$

$$f'(p) = \lim_{t \to p} \frac{f(t)-f(p)}{t-p} = \lim_{t \to p^+} \frac{f(t)-f(p)}{t-p} \overset{②}{\le} 0$$

(Same lemma)

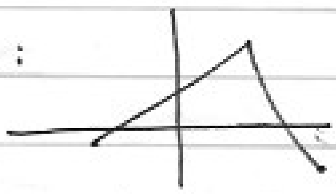
$$0 \overset{①}{\le} f'(p) \overset{②}{\le} 0$$

Hence $f'(p) = 0$. #

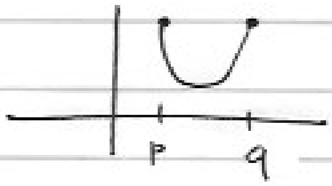
We proved

$$\left.\begin{array}{l} p \text{ interior} \\ p \text{ local max (or min)} \\ f'(p) \text{ exist} \end{array}\right\} \Longrightarrow f'(p) = 0.$$

Caution:



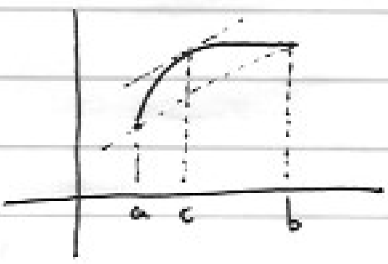
local max
$f'(p)$ DNE



$f'(p) < 0$
$f'(q) > 0$

MVT (Mean Value Thm)

Let $f : [a,b] \longrightarrow \mathbb{R}$, $a < b$ s.t.
(i) f is continuous on $[a,b]$, and
(ii) f is differentrable on (a,b).
Then $\exists\, c \in (a,b)$ s.t



$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

(GMVT) Generalized Mean Value Thm.

Let $f, g : [a,b] \longrightarrow \mathbb{R}$, $a < b$, s.t.
Both f and g are continuous on $[a,b]$, and both f and g are differentiable on (a,b), then $\exists\, c \in (a,b)$ s.t.

$$\big(f(b) - f(a)\big)g'(c) = \big(g(b) - g(a)\big)f'(c).$$

Obs GMVT $\Longrightarrow$ MVT. $(g(x) = x)$

Proof:

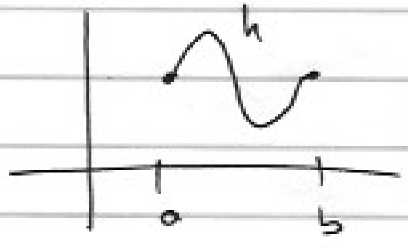
Let $h(t) = \big(f(b) - f(a)\big)g(t) - \big(g(b) - g(a)\big)f(t)$

h is continuous on all of $[a,b]$ since f & g are
h is diff'ble on all of (a,b) since f & g are.

$$\begin{aligned} h(a) &= \big(f(b) - f(a)\big)g(a) - \big(g(b) - g(a)\big)f(a) \\ &= f(b)g(a) - g(b)f(a) \end{aligned}$$

$$\begin{aligned} h(b) &= \big(f(b) - f(a)\big)g(b) - \big(g(b) - g(a)\big)f(b) \\ &= -f(a)g(b) + g(a)f(b) \end{aligned}$$

$h(a) = h(b)$.

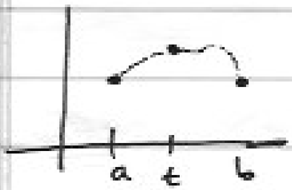


Claim $\exists\, c \in (a,b)$ s.t. $h'(c) = 0$.

Case 1 $\quad h$ is constant on $[a,b]$.

$\Rightarrow \quad h'(x) \equiv 0$ on (a,b)

Case 2 $\quad h(t) > h(a)$ for some $t \in (a,b)$



$\max\limits_{[a,b]} h = h(c) > h(a) = h(b)$, $c \in [a,b]$

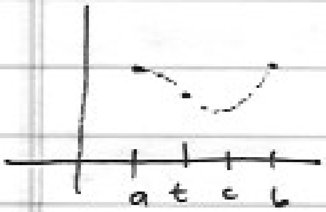
by Extreme Value Thm.

$c \neq b, a$.

$c \in (a,b)$

$h'(c) = 0$. 1^{st} DT.

Case 3 $\quad h(t) < h(a)$ for some $t \in (a,b)$



$\min\limits_{[a,b]} h = h(c) < h(a) = h(b)$, $c \in [a,b]$.

by EVT.

$c \neq a, b$

$c \in (a,b)$

$h'(c) = 0 \quad 1^{st}$ DTest

$$h'(t) = (f(b) - f(a))\, g'(t) - (g(b) - g(a))\, f'(t)$$

$$0 = h'(c) = (f(b) - f(a))\, g'(c) - (g(b) - g(a))\, f'(c).$$

#

<u>Corollary</u> Let $f:(a,b)\to\mathbb{R}$ be diff'ble

(a) $\underset{\text{on } (a,b)}{f'(x)\geq 0} \iff \forall x,y\in(a,b)\ \big(x\leq y \Rightarrow f(x)\leq f(y)\big)$

(b) $\underset{\text{on } (a,b)}{f'(x)\leq 0} \iff \forall x,y\in(a,b)\ \big(x\leq y \Rightarrow f(x)\geq f(y)\big)$

(c) $\underset{\text{on } (a,b)}{f'(x)\equiv 0} \iff f\equiv \text{constant on } (a,b).$

(a) ($\Rightarrow$:) $\forall x,y \quad a<x<y<b$

by MVT $\quad \exists\, c\in(x,y)$ s.t. $f'(c)=\dfrac{f(y)-f(x)}{y-x}\geq 0$

$f(y)-f(x)\geq 0$ since $y-x>0$

$f(y)\geq f(x).$

(a) ($\Leftarrow$:) $\forall x,y\in(a,b)$

$x<y \Rightarrow f(x)\leq f(y)$

not MVT.

Lemma from April 15

$$0\leq \frac{f(x)-f(y)}{x-y}$$

$$0\leq \lim_{y\to x^+}\frac{f(x)-f(y)}{x-y}=f'(x)$$

since f is diff'ble.

Still <u>True</u> (a') $\underset{\text{on } (a,b)}{f'(x)>0} \Rightarrow \big(x<y \Rightarrow f(x)<f(y)) \text{ on } (a,b)\big)$

$\exists\, c\in(x,y) \quad 0<f'(c)=\dfrac{f(y)-f(x)}{y-x} \quad (x<y)$

$\Rightarrow f(y)-f(x)>0 \quad \#$

(a')($\Leftarrow$:) is false:

$$\left(\forall x,y \in (a,b),\ x<y \Rightarrow f(x)<f(y)\right)$$

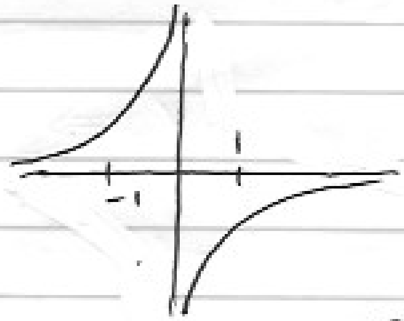
$$\not\Rightarrow f'(x)>0 \text{ on } (a,b).$$

Ex $f(x)=x^3$ on $(-1,1)$

$(x<y \Rightarrow x^3<y^3)$ but $f'(0)=0$.

Caution In corollary, it is important the domain $=(a,b)$, which is connected.

Ex (a) $f(x)=\frac{-1}{x}$, $f'(x)=\frac{+1}{x^2}>0$.



locally increasing but not globally:

$-1<1$ but $\overbrace{f(-1)}^{1} \not< \overbrace{f(1)}^{-1}$

Domain $(-\infty,0)\cup(0,\infty)$ is not connected

(b) $h(x)=\begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases} : \mathbb{R}-\{0\} \to \mathbb{R}$

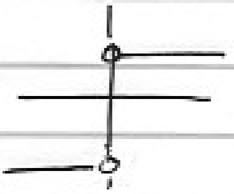
$h'(x)\equiv 0$, domain h = domain $h'=(-\infty,0)\cup(0,\infty)$

h is not constant globally.

Darboux Thm Let $f:[a,b] \longrightarrow \mathbb{R}$ be differentiable on all of $[a,b]$.
If $f'(a) < \lambda < f'(b)$ then there exists $c \in (a,b)$ s.t. $f'(c) = \lambda$.

Caution Regardless of f' being continuous or not, f' satisfies intermediate value property.
This is **NOT** Intermediate Value Theorem applied to f'.

Example (i) $g(x) = |x|$, $g'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases} = h(x)$

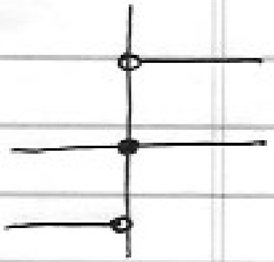


This is a derivative

Darboux Thm doesnot apply since g is not diffble at 0.

(ii) Is there a function f s.t.

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0. \\ -1 & \text{if } x < 0 \end{cases}$$



This can't be a derivative.

Answer is NO. f' is supposed to satisfy intermediate value property. This function doesn't

Proof of Darboux Thm

Given $f: [a,b] \to \mathbb{R}$ differentiable on all of $[a,b]$.

Given $f'(a) < \lambda < f'(b)$.

Let $g(t) = f(t) - \lambda t$ diff'ble on $[a,b]$.

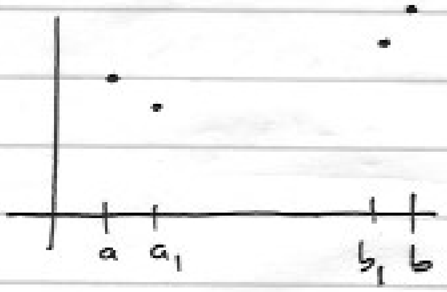
$$g'(t) = f'(t) - \lambda$$
$$g'(a) = f'(a) - \lambda < 0.$$

$$g'(a) = \lim_{t \to a^+} \frac{g(t) - g(a)}{t - a} < 0$$

(Lemma 4/15) $\Rightarrow \exists \delta > 0 \ \forall t \quad a < t < a + \delta \quad \frac{g(t) - g(a)}{t - a} < 0$

Take any $a_1 \in (a, a+\delta)$, $\frac{g(a_1) - g(a)}{a_1 - a} < 0$, $a_1 - a > 0$

$\Rightarrow g(a_1) < g(a)$.



$$g'(b) = f'(b) - \lambda > 0$$
$$g'(b) = \lim_{t \to b^-} \frac{g(t) - g(b)}{t - b} > 0$$

(Lemma 4/15) $\exists \delta' > 0, \forall t \quad b - \delta' < t < b \quad \frac{g(t) - g(b)}{t - b} > 0$

Take any $b_1 \in (b - \delta', b)$, $\frac{g(b_1) - g(b)}{b_1 - b} > 0$, $b_1 - b < 0$

$$g(b_1) < g(b)$$

Extreme Value Thm $\Rightarrow \min_{[a,b]} g$ exists $= g(c)$ for some $c \in [a,b]$.

$c \neq a, b$; since $g(c) \leq g(a_1) < g(a)$, $g(c) \leq g(b_1) < g(b)$

$g'(c) = 0 = f'(c) - \lambda$, since $c \in (a,b)$. (1st DTest)

$$f'(c) = \lambda. \quad \#$$