April 15, 2020

ANNOUNCEMENT



Midterm 2

Take home, on Chapters III and IV, essentially

- To be posted on April 22, Wednesday 5pm on ICON 4210 0A01, files
- Due April 24, by midnight
- To be uploaded to ICON MATH 4210 0A01

Exam should take 1.5 hrs to write

6-7 problems, including 1 true/false question.

No definitions

Please:

- Write clearly and not too small

(scanning/copying small print can be a problem)

- Use complete sentences.
- Everything must be in logical order.
- Show details.

Since it is take-home, you have a little more time to do that.

You can use our class notes and textbook.

- No external sources, such as other books, notes.
- No online sources.
- Cannot communicate with other students about the test and the solutions.

# CHAPTER V DIFFERENTIATION

Defn Let $f: [a,b] \longrightarrow \mathbb{R}$. For a given $x \in [a,b]$ one defines

$$\phi(t) = \frac{f(t) - f(x)}{t - x} \quad \text{for} \quad a < t < b,\ t \neq x$$

( $x$ is fixed and $\phi$ depends on $x$ )

$\lim_{t \to x} \phi(t) = f'(x)$ if it exists.

$f'(x)$ is called the derivative of $f$ at $x$ when it is defined.

If $f'(x)$ exists, then $f$ is called differentiable at $x$.

$f$ is called differentiable if it is differentiable at each $x \in [a,b] = \text{domain}(f)$.

Domain $f' \subseteq$ Domain $f$

Thm 5.2 Let $f: [a,b] \longrightarrow \mathbb{R}$, $x \in [a,b]$.
$f$ is diff'ble at $x \implies f$ is continuous at $x$.

Proof: $f(t) - f(x) = \dfrac{f(t) - f(x)}{t - x} \cdot (t - x) \cdot$ for $t \neq x$

as $t \to x$ $\quad \dfrac{f(t) - f(x)}{t - x} \downarrow f'(x) \qquad (t-x) \downarrow 0$

$$\lim_{t \to x} f(t) - f(x) = f'(x) \cdot 0 = 0 \quad (\text{Prop } 4.4)$$

$$\lim_{t \to x} f(t) = f(x)$$

$f$ is continuous at $x$. (prop 4.6)

<u>Thm 5.3</u> Let $f, g : [a,b] \to \mathbb{R}$, and $f$ and $g$ be diff'ble at $x$. Then, $f+g$, $fg$, $f/g$ ($g(x) \neq 0$), are diff'ble at $x$ and

a) $(f+g)'(x) = f'(x) + g'(x)$

b) $(fg)'(x) = f'(x)\, g(x) + f(x) \cdot g'(x)$

c) $\left(\frac{f}{g}\right)'(x) = \dfrac{f'(x)\, g(x) - f(x)\, g'(x)}{(g(x))^2}$ (if $g(x) \neq 0$)

<u>Proof of (b)</u> Let $h(x) = f(x)\, g(x)$

$$h(t) - h(x) = f(t)\, g(t) - f(x)\, g(x)$$

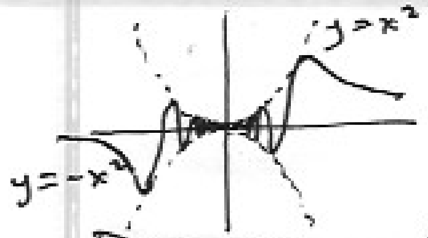
$$= f(t)\big(g(t) - g(x)\big) + g(x)\big(f(t) - f(x)\big)$$

$$\lim_{t \to x} \frac{h(t) - h(x)}{t - x} = \lim_{t \to x} f(t) \cdot \frac{g(t) - g(x)}{t - x} + \lim_{t \to x} g(x) \frac{f(t) - f(x)}{t - x}$$

$= f(x) \cdot g'(x) + g(x) \cdot f'(x)$ exists.

RHS exists
$\Rightarrow$ LHS exists.

and hence equals to $h'(x)$.




<u>Example</u> $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

- $f'(0) = \lim_{t \to 0} \frac{f(t) - f(0)}{t - 0} = \lim_{t \to 0} \frac{t^2 \sin \frac{1}{t} - 0}{t} = \lim_{t \to 0} t \sin \frac{1}{t}$
$= 0$

since $\forall \varepsilon > 0 \ \exists \delta = \varepsilon \ \forall t, \ 0 < |t - 0| < \delta$
$\Rightarrow \left| t \sin \frac{1}{t} - 0 \right| = |t| \underbrace{\left| \sin \frac{1}{t} \right|}_{\leq 1} \leq |t| < \delta = \varepsilon.$

- If $x \neq 0$ $f'(x) = 2x \sin \frac{1}{x} + x^2 \left( -\frac{1}{x^2} \right) \cos \frac{1}{x}$

$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

1. $f$ diff'ble on all of $\mathbb{R}$
2. $f$ continuous on all of $\mathbb{R}$
3. $f'$ is not continuous at 0. Why?

$\lim_{x \to 0} f'(x) \neq f'(0) = 0$, since $\lim_{x \to 0} f'(x)$ DNE:

Choose $t_n = \frac{1}{2\pi n.} \to 0$, $\cos \frac{1}{t_n} = \cos 2\pi n = 1 \to 1$

$s_n = \frac{1}{(2n+1)\pi} \to 0$, $\cos \frac{1}{s_n} = \cos (2n+1)\pi = -1 \to -1$

use Thm 4.2

$2 \underset{\downarrow\, 0}{s_n} \underset{\text{bounded}}{\sin \frac{1}{s_n}} - \underset{\downarrow\, -1}{\cos \frac{1}{s_n}} \to +1$, $2 \underset{\downarrow\, 0}{t_n} \underset{\text{bdd.}}{\sin \frac{1}{t_n}} - \underset{\downarrow\, 1}{\cos \frac{1}{t_n}} \to -1$

Compare $g(x) = |x|$, $g'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$

$g'(0)$ DNE

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$g$ is not diff'ble at $0$.
$g'(x)$ is not defined at $0$
domain$(g'(x)) = \mathbb{R} - \{0\}$
$g'(x)$ is continuous on its domain. $\mathbb{R} - \{0\}$
Since $0 \notin$ domain$(g')$, talking about the continuity of $g'$ at $0$ is not meaningful.

But
$f'(0)$ is defined,
$0 \in$ domain$(f') = \mathbb{R}$
"$f'$ is not continuous at $0$" makes sense.
and it is a correct statement

## 5.5 CHAIN RULE

Let $f : [a,b] \longrightarrow \mathbb{R}$, $f([a,b]) \subseteq I$ interval
$g : I \longrightarrow \mathbb{R}$.

If $f$ is diff'ble at $c \in [a,b]$, and
$g$ is diff'ble at $f(c) = d$,
then $h(t) = g(f(t))$ is differentiable at $c$, and
$$h'(c) = g'(f(c)) \cdot f'(c).$$

Remark: This is a stronger theorem than seen in calculus books, since the assumption of differentiability is at one pt for each of $f$ and $g$.

Proof:

Let $\dfrac{f(t) - f(c)}{t - c} - f'(c) = u(t)$ for $t \neq c$.

$f$ diff'ble at $c$

① $f(t) - f(c) = (t - c)(f'(c) + u(t))$

as $t \to c$ $u(t) \longrightarrow 0$ since $\dfrac{f(t) - f(c)}{t - c} \to f'(c)$

$d = f(c)$: Let $\dfrac{g(s) - g(d)}{s - d} - g'(d) = v(s)$ for $s \neq d$

② $g(s) - g(d) = (s - d)(g'(d) + v(s))$

as $s \to d$, $v(s) \longrightarrow 0$ since $\dfrac{g(s) - g(d)}{s - d} \to g'(d)$

Observe that ① is true if $t \neq c$, by def$^n$ of $u$.
① is true if $t = c$, both sides are 0
provided that $u(c)$ is defined
and we take $u(c) = 0$.

Similarly ② is true if $s \neq d$, by def$^n$ of $v$
② is true if $s = d$, both sides are 0
provided that $v(d)$ is defined
and we take $v(d) = 0$.

$$h(t) = g(f(t))$$

$$h(t) - h(c) = g(\overbrace{f(t)}^{s}) - g(\overbrace{f(c)}^{d})$$

$$\overset{\text{by } ②}{=} (f(t) - f(c))(g'(d) + v(f(t)))$$

$$\overset{\text{by } ①}{=} (t-c)(f'(c) + u(t))(g'(d) + v(f(t)))$$

If $t \neq c$, $\dfrac{h(t) - h(c)}{t-c} = (f'(c) + u(t))(g'(d) + v(f(t)))$

as $t \to c$
$u(t) \to 0$

$f'(c)$

$f$ diff'ble at $c$
$\Rightarrow$
$f$ continuous at $c$
as $t \to c$
$f(t) \to f(c) = d$
$v(f(t)) \to 0$

$g'(d)$

$$h'(c) = f'(c)\, g'(d) = f'(c)\, g'(f(c)).$$

RHS converges $\Rightarrow$ LHS converges. $h'(c)$ exists.

**Lemma:** Let $f: I \longrightarrow \mathbb{R}$, $b \in I$ and $\lim_{x \to b} f(x) = a$ exists, $a \in \mathbb{R}$.

(i) If $\lim_{x \to b} f(x) > c$,

then $\exists \delta > 0 \ \forall x, 0 < |x-b| < \delta, \ f(x) > c$

(ii) If $\lim_{x \to b} f(x) < c$,

then $\exists \delta > 0 \ \forall x, \ 0 < |x-b| < \delta \Rightarrow f(x) < c$

(iii) If $\exists \delta > 0$ s.t. $\forall x, \ 0 < |x-b| < \delta \Rightarrow f(x) \geqslant c$

then $\lim_{x \to b} f(x) \geqslant c$

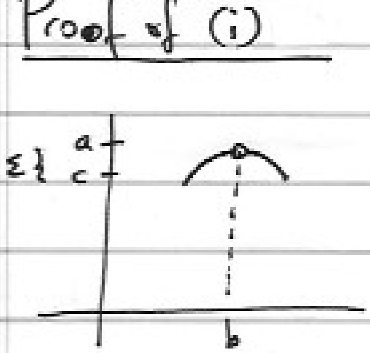
Caution (i) is false with $\geqslant$ (or $\leqslant$ in (ii))

$f(x) = x^3 \quad \lim_{x \to 0} x^3 = 0 \geqslant 0$ but $x^3 < 0$ for $x < 0$

(iii) is false with $>$

$f(x) = x^2, \ \underset{\text{if } x \neq 0}{x^2 > 0}, \ \lim_{x \to 0} x^2 = 0 \not> 0$

Proof of (i)




$a > c$, Let $\varepsilon = a - c > 0$

then $\exists \delta > 0$

$0 < |x-b| < \delta \Rightarrow |f(x) - a| < \varepsilon$

$c - a < f(x) - a < a - c$

$c < f(x)$.

(ii) is done similarly

(iii) Assume hypothesis

$\exists \delta > 0$ s.t. $\forall x$, $0 < |x-b| < \delta \Rightarrow f(x) \geq c$

Suppose $\liminf_{x \to b} f(x) < c$.

Then by (ii) $\exists \delta' > 0$ $\forall x$, $0 < |x-b| < \delta'$
$f(x) < c$

For $\delta'' = \min(\delta, \delta') > 0$
$\forall x$, $|x-b| < \delta''$ we will have
both $f(x) < c$ and $f(x) \geq c$

Contradiction.