# CONTINUITY and CONNECTEDNESS

***** **Theorem 4.22**

Let $f:(X,d_X) \to (Y,d_Y)$ be continuous. If $E$ is a connected subset of $X$, then $f(E)$ is a connected subset of $Y$.

**Recall** 1) $A, B \subseteq X$, $A$ and $B$ are called separated if $\bar{A} \cap B = A \cap \bar{B} = \emptyset$.

2) A set $E \subseteq X$ is called connected if there exists no $A, B \subseteq X$ s.t.

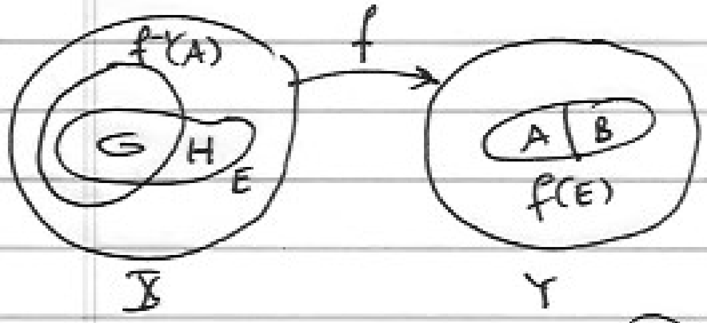
(i) $A \neq \emptyset$, $B \neq \emptyset$,

(ii) $E = A \cup B$, and

(iii) $\bar{A} \cap B = A \cap \bar{B} = \emptyset$.

**Proof of Thm 4.22** Proof by Contrapositive

We start with a non-trivial separation of $f(E)$ to find a non-trivial separation of $E$.




Suppose $\exists\ A, B \subseteq Y$ s.t.

(i) $A \neq \emptyset$, $B \neq \emptyset$

(ii) $f(E) = A \cup B$

(iii) $\bar{A} \cap B = A \cap \bar{B} = \emptyset$.

Define $G = f^{-1}(A) \cap E$

$H = f^{-1}(B) \cap E$.

We want to establish (i) $G \neq \phi$, $H \neq \phi$

(ii) $E = G \cup H$

(iii) $\bar{G} \cap H = G \cap \bar{H} = \phi$.

(i) $A \neq \phi$ and $A \subseteq f(E)$

$\exists\, a \in A$, $a = f(c)$ for some $c \in E$.

Hence $\exists\, c \in f^{-1}(A) \cap E = G$.

$G \neq \phi$. Similarly $H \neq \phi$.

(ii)
$$
\begin{aligned}
G \cup H &= \left(f^{-1}(A) \cap E\right) \cup \left(f^{-1}(B) \cap E\right) \\
&= \left(f^{-1}(A) \cup f^{-1}(B)\right) \cap E \\
&= f^{-1}(A \cup B) \cap E \\
&= f^{-1}(f(E)) \cap E = E
\end{aligned}
$$

Obs.

$x \in E \Rightarrow f(x) \in f(E)$
$\Rightarrow x \in f^{-1}(f(E))$

Hence $E \subseteq f^{-1}(f(E))$ (Caution: this is not always $=$)

(iii) $A \subseteq \bar{A}$

$$G = f^{-1}(A) \cap E \subseteq f^{-1}(A) \subseteq \underbrace{f^{-1}(\bar{A})}_{\text{closed since } \bar{A} \text{ is closed, } f \text{ continuous}}$$

$\bar{G} \subseteq f^{-1}(\bar{A})$ (prop. 2.27c)

$$\bar{G} \cap H \subseteq f^{-1}(\bar{A}) \cap f^{-1}(B) = f^{-1}(\bar{A} \cap B) = f^{-1}(\phi) = \phi.$$

Similarly $G \cap \bar{H} = \phi$.

Hence $G, H$ provide a non-trivial separation of $E$.

<u>Recall Thm 2.47</u> Let $E \subseteq \mathbb{R}$

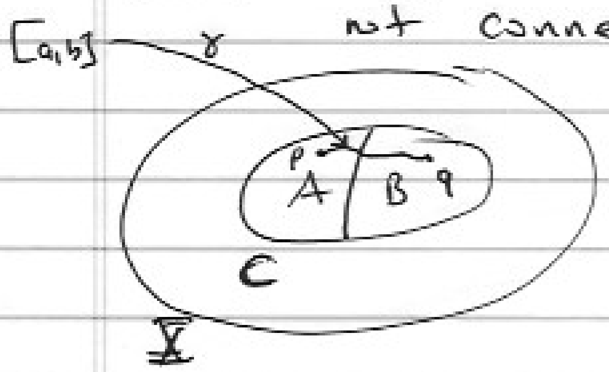
$$E \text{ is connected} \iff \overbrace{\forall x,y \in E\ \forall z \in \mathbb{R}\ (x<z<y \Rightarrow z \in E)}^{E \text{ is an interval}}$$

Not in the book ↓

<u>Def'n</u> A subset $C \subseteq (X,d)$ is called path-connected if $\forall p,q \in C$, $\exists \gamma : [a,b] \to C$ s.t. $\gamma$ is continuous and $\gamma(a) = p$, $\gamma(b) = q$.

<u>Prop 1</u> ($C$ is path-connected $\Rightarrow$ $C$ is connected) for $C \subseteq (X,d)$ any metric space.

<u>Proof</u> Suppose $C$ is path-connected but not connected.




Suppose $\exists A, B \subseteq X$ s.t.
(i) $A \neq \emptyset$, $B \neq \emptyset$
(ii) $A \cup B = C$
(iii) $\overline{A} \cap B = A \cap \overline{B} = \emptyset$

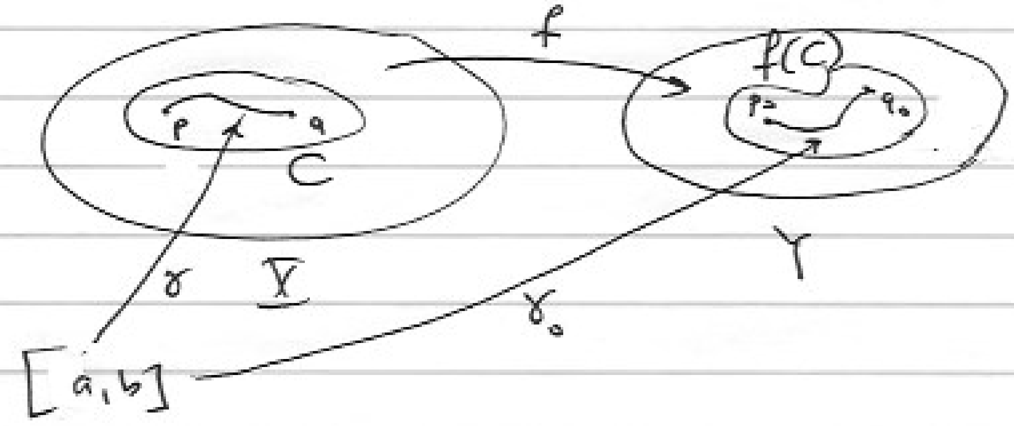
$\exists p \in A$, $\exists q \in B$. Path-connected $C$
$\Rightarrow \exists \gamma : [a,b] \to C$ s.t.
$\gamma$ is continuous, $\gamma(a) = p$, $\gamma(b) = q$.

The same way we proved Thm. 4.22:
Take $G = \gamma^{-1}(A)$ (in $[a,b]$)
$H = \gamma^{-1}(B)$, and show that
$G$ & $H$ provide a non-trivial separation of $[a,b]$. But $[a,b]$ is connected.
Contradiction.

**Prop 2** Let $f:(X,d_X)\to(Y,d_Y)$ be continuous. For all $C\subseteq X$, if $C$ is path-connected, then $f(C)$ is path-connected.

**Proof:** Given $p_0, q_0 \in f(C)$.

$\exists\, p,q\in C$ s.t. $f(p)=p_0$ & $f(q)=q_0$




$C$ is path-connected: $\exists\, \gamma:[a,b]\to C$ s.t. $\gamma$ is continuous, $\gamma(a)=p$, $\gamma(b)=q$.

Let

$\gamma_0 = f\circ\gamma : [a,b]\to f(C)$,

$\gamma_0(a)=f(\gamma(a))=f(p)=p_0$

$\gamma_0(b)=f(\gamma(b))=f(q)=q_0$

$\gamma_0=f\circ\gamma$ is continuous. Hence $\forall\, p_0,q_0\in f(C)$ can be connected with a path in $f(C)$.

**Without Proof:** (Not in the test)

**Thm:** $A\subseteq \mathbb{R}^k$, $A$ open, connected (Standard metric) $\Rightarrow$ $A$ is path-connected.

**Ex:** $C=\{(0,y)\mid -1\le y\le 1\}\cup\{(x,\sin\frac{1}{x})\mid x>0\}\subseteq\mathbb{R}^2$ (standard metric)

$C$ is connected but not path connected.

not in the book.

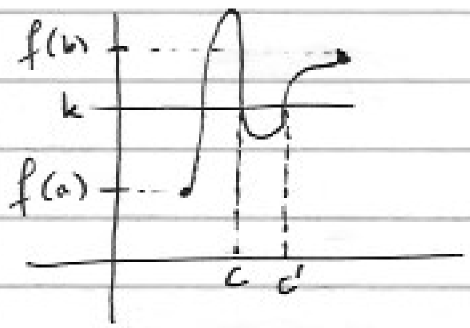
4.23 **Theorem:** Intermediate Value Thm

Let $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be continuous (Standard metric)

If $k \in \mathbb{R}$ s.t. either $f(a) < k < f(b)$

or $f(b) < k < f(a)$,

then $\exists\, c \in (a,b)$ s.t. $f(c) = k$.




**Proof.** $[a,b]$ is connected.

Thm 2.47

$J = f([a,b])$ is connected

Thm 4.22

$f(a), f(b) \in J$.

If $k$ is between $f(a)$ and $f(b)$, then $k \in J = f([a,b])$

(Thm 2.47)

$k = f(c) \in f([a,b])$, for some $c \in [a,b]$.

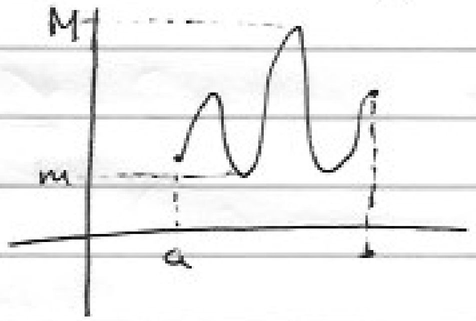
$c \neq a, b$ since $k \neq f(a), f(b)$.

$c \in (a,b)$.

**Corollary:** Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be continuous.

(i) $I$ is an interval $\Rightarrow f(I)$ is an interval

(ii) If $I = [a,b]$, then $f(I) = f([a,b]) = [m, M]$.




Using Extreme Value Thm.

Remark: IVT (and EVT) $\Rightarrow$ (i) and (ii). We prove here.
Also (i) $\Rightarrow$ IVT, and (ii) $\Rightarrow$ IVT. (HW)

Proof of Corollary*. (By IVT & EVT)

(i) Let $k_1, k_2 \in f(I)$ be given, s.t. $k_1 < k_2$.
Assume $k_1 < k < k_2$, for an arbitrary $k \in \mathbb{R}$
$\exists\, a_1, a_2 \in I \quad f(a_1) = k_1$
$f(a_2) = k_2$
By IVT $\Rightarrow \exists\, c$ between $a_1$ and $a_2$ s.t.
$f(c) = k$.
$a_1, a_2 \in I$; $I$ interval : $c \in I$.
$k = f(c) \in f(I)$.
Hence $\forall k_1, k_2 \in f(I)\ \forall k \in \mathbb{R}$
$k_1 < k < k_2 \Rightarrow k \in f(I)$.
$f(I)$ is an interval.

(ii) $I = [a,b] \quad f: I \to \mathbb{R}$ continuous.
$[a,b]$ compact $\Rightarrow$ $f$ attains its max and min values on $[a,b]$; $\exists\, p, q \in [a,b]$ s.t.
$\forall x \in [a,b] \quad f(p) = m \le f(x) \le M = f(q)$
EVT (By Extreme Value Thm.)
$\Rightarrow f([a,b]) \subseteq [m, M]$. ①
$\forall k \in [m, M], \quad f(p) \le k \le f(q)$
$\Rightarrow \exists\, c$ either between $p$ and $q$
or $p = c$ or $q = c$ s.t.
$f(c) = k \in f([a,b])$ by IVT.
$\forall k \in [m,M], \quad k \in f([a,b])$
$[m, M] \subseteq f([a,b])$ ② Hence "="

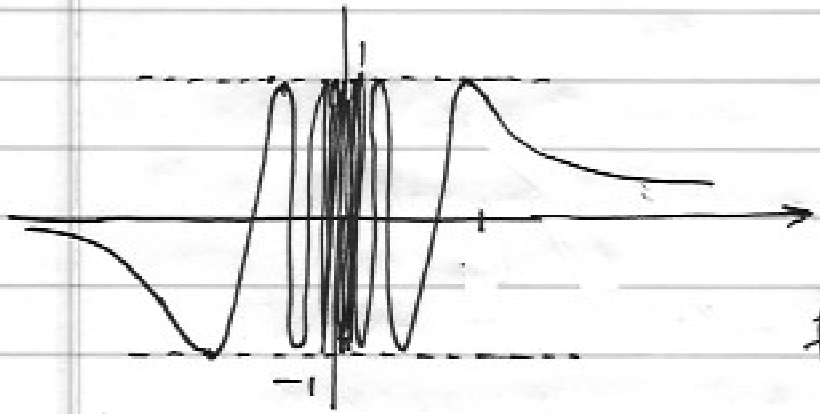
<u>Caution</u>: Other than compact intervals $[a,b]$, $I$ and $f(I)$ may be different type of intervals.

<u>Ex</u> $f(x) \equiv 0$

$$f([1,\infty)) = [0,0].$$
$$f((-2,5)) = [0,0].$$

<u>Ex</u>

$$f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$




$$\lim_{x\to 0}\left(\sin\frac{1}{x}\right) = \text{DNE}$$

$$f(0) = 0$$

$f$ is not continuous at $0$,

but $f$ satisfies intermediate Value Property:

$$\forall a,b \in \mathbb{R}, \text{ if } f(a) < k < f(b), \text{ then } \exists c \in (a,b) \text{ s.t. } f(c) = k.$$

Why? $\forall \varepsilon > 0$ $f([-\varepsilon,\varepsilon]) = [-1,1]$.

For $a,b > 0$ $f$ is continuous between $a$ & $b$.

$a,b < 0$ " " " " $a$ & $b$.