Thm 4.17 Let $f:(X, d_X) \longrightarrow (Y, d_Y)$ be

- one-to-one and onto, and
- continuous.

Let $g:(Y, d_Y) \longrightarrow (X, d_X)$ be defined by

$$g(y) = x \iff f(x) = y \quad (\text{i.e. } g = f^{-1})$$

If X is compact, then g is continuous.

Proof: Want to show $\forall A$ closed in X,
$g^{-1}(A)$ is closed Y.
(use corollary of Thm 4.8 p 87)

Let A be a closed subset of X
A closed $\subseteq X$ compact.
A is compact (in X) Thm 2.35
$f(A)$ is compact (in Y) Thm 4.14.
$f(A)$ is closed in Y Thm 2.34.

$f(A) = g^{-1}(A)$ since:
$f(A) = \{f(x) \in Y \mid x \in A\} \subseteq Y$
$g^{-1}(A) = \{y \in Y \mid g(y) \in A\} \subseteq Y$
Recall $g(y) = x \iff f(x) = y$

$g^{-1}(A)$ closed in Y
g is continuous, on Y.

Compare Thm 4.17 to example from 4/8/20

$f: [0, 2\pi) \to S^1$

$g = f^{-1}$ is not continuous

$[0, 2\pi)$ is not compact.

UNIFORM CONTINUITY

Defn Let $f: (X, d_X) \to (Y, d_Y)$.

f is called uniformly continuous on X if

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x, y \in X,$

$$d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon.$$

Compare to continuous on X:

$\forall x \in X \ \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall y \in X$

$$d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$$

δ may depend on x & ε in continuity

δ is independet of x, but depends on ε in uniform continuity.

Obs: Uniform Continuity $\Rightarrow$ continuity

Ex. Continuity $\not\Rightarrow$ uniform Continuity

Thm. Compact X + Continuity $\Rightarrow$ uniform Continuity

Examples :

Lipschitz function.

① If $f(x): \mathbb{R} \to \mathbb{R}$ satisfies

$\exists M \in \mathbb{R}\ \forall x, y \in \mathbb{R} \quad |f(x) - f(y)| \leq M|x-y|$,

then f is uniformly continuous:

Reason: $M \geq 0$, $M+1 > 0$.

$\forall \varepsilon > 0\ \exists \delta = \varepsilon/(M+1) > 0$

$\forall x, y \in \mathbb{R} \quad |x-y| < \delta \Rightarrow |f(x) - f(y)| \leq M|x-y|$

$$\leq M\delta = \frac{M\varepsilon}{M+1} < \varepsilon$$

② $f(x,y): \mathbb{R}^2 \to \mathbb{R}^1$ (standard metrics)

$$f(x,y) = Ax + By + C, \quad A, B, C \in \mathbb{R}.$$

Then f is uniformly continuous:

$$\forall \varepsilon > 0 \quad \exists \delta = \frac{\varepsilon}{|A| + |B| + 1} > 0$$

$\forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$

$\|(x_1, y_1) - (x_2, y_2)\| < \delta \Longrightarrow$

$$|f(x_1, y_1) - f(x_2, y_2)| = |Ax_1 + By_1 + C - Ax_2 + By_2 + C|$$

$$= |A(x_1 - x_2) + B(y_1 - y_2)|$$

$$\leq |A|\,|x_1 - x_2| + |B|\,|y_1 - y_2|$$

$$\leq (|A| + |B|)\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$< (|A| + |B|) \cdot \delta$$

$$= \frac{|A| + |B|}{|A| + |B| + 1}\varepsilon < \varepsilon.$$

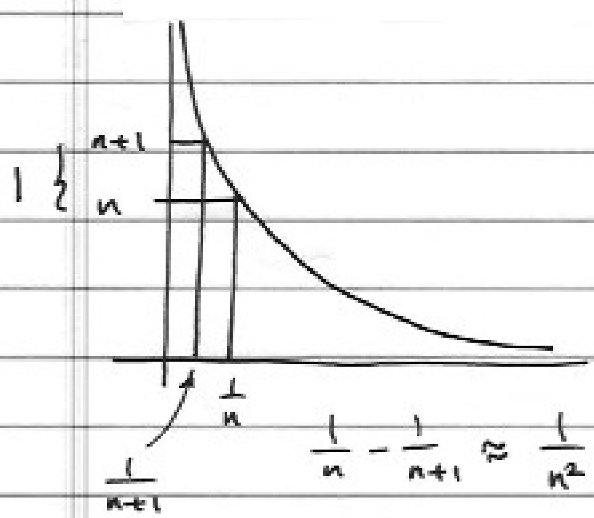
Ex 3 $f(x) = \frac{1}{x} : (0,\infty) \to (0,\infty)$ is NOT uniformly continuous.

Want to show

$$\text{not}\left(\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x, y \in (0,\infty) \quad |x-y| < \delta \Rightarrow \left|\frac{1}{x} - \frac{1}{y}\right| < \varepsilon \right)$$

Want to show

$$\exists \varepsilon > 0 \ \forall \delta > 0 \ \exists x_\delta, y_\delta \in (0,\infty) \quad |x_\delta - y_\delta| < \delta \text{ and } \left|\frac{1}{x_\delta} - \frac{1}{y_\delta}\right| \geq \varepsilon$$



$$\frac{1}{n} - \frac{1}{n+1} \approx \frac{1}{n^2}$$

$$\exists \varepsilon = 1 \ \forall \delta > 0 \ \exists n \in \mathbb{N} \quad \frac{1}{n^2} \leq \delta$$

$$\exists x_n = \frac{1}{n}, \ y_n = \frac{1}{n+1}$$

$$|x_n - y_n| = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} < \frac{1}{n^2} \leq \delta, \text{ but}$$

$$\left|\frac{1}{x_n} - \frac{1}{y_n}\right| = |n - (n+1)| = 1 \geq \varepsilon$$

HW Show $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

*** Theorem 4.19

If $f:(X, d_X) \to (Y, d_Y)$ is continuous,
and X is compact,
then f is uniformly continuous on X.

Proof

Let f be continuous on X.
X is compact.
Let $\varepsilon > 0$ be given.

For every $p \in X$, by continuity of f at p

① $\exists \delta_p > 0$ s.t. $\forall q \in X$
$d_X(p,q) < \delta_p \Rightarrow d_Y(f(p), f(q)) < \frac{\varepsilon}{2}$.

② Let $V_p = \{ q \in X \mid d_X(p,q) < \frac{\delta_p}{2} \}$.

$p \in V_p$ and V_p is open in X.

$X \subseteq \bigcup_{p \in X} V_p$. X is compact.

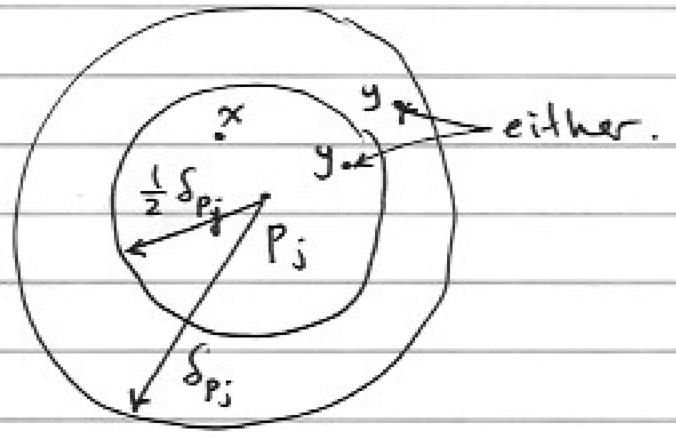
$\Rightarrow \exists p_1, p_2, \ldots p_m \in X$ s.t. $X \subseteq \bigcup_{i=1}^{m} V_{p_i} \subseteq X$

③ Let $\delta = \frac{1}{2} \min (\delta_{p_1}, \delta_{p_2}, \ldots \delta_{p_m}) > 0$.

This is δ we want for ε on all of X:

Let $x, y \in \mathbb{X}$ s.t. $d_{\mathbb{X}}(x,y) < \delta$.

$x \in \mathbb{X} = \bigcup_{i=1}^{m} V_{p_i}$. $x \in V_{p_j}$ for some j, $1 \le j \le m$.



$d_{\mathbb{X}}(x, p_j) < \frac{\delta_{p_j}}{2}$ (defn of V_{p_j}) by ②

$$d_{\mathbb{X}}(y, p_j) \le d_{\mathbb{X}}(y, x) + d_{\mathbb{X}}(x, p_j)$$

$$< \delta + \frac{\delta_{p_j}}{2} \underset{③}{\le} \frac{\delta_{p_j}}{2} + \frac{\delta_{p_j}}{2} = \delta_{p_j}$$

So $d_{\mathbb{X}}(x, p_j) < \delta_{p_j}$ and $d(y, p_j) < \delta_{p_j}$

$d_Y(f(x), f(p_j)) < \frac{\varepsilon}{2}$, $d_Y(f(y), f(p_j)) < \frac{\varepsilon}{2}$ by ①

$$d_Y(f(x), f(y)) \le d_Y(f(x), f(p_j)) + d_Y(f(p_j), f(y))$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence

$\forall \varepsilon > 0, \exists \delta > 0, \forall x, y, d_{\mathbb{X}}(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$.

#

Example

Let (X, d) be any metric space.
Fix $p_0 \in X$

define
$$f : X \to [0, \infty) \text{ by } f(q) = d(p_0, q)$$

$\forall \varepsilon > 0 \ \exists \delta = \varepsilon > 0$

$\forall p, q \in X, \ d(p,q) < \delta \Rightarrow$

$$|f(p) - f(q)| = |d_X(p, p_0) - d_X(q, p_0)|$$
reverse Δ-ineq.
$$\leq d(p,q) < \delta = \varepsilon$$

So f lipschitz and hence uniformly continuous.

Example Let (X, d_0) be the discrete metric space

$$d(p,q) = \begin{cases} 1 & \text{if } p \neq q \\ 0 & \text{if } p = q. \end{cases}$$

For any function $f : (X, d_0) \to (Y, d_Y)$

$\forall \varepsilon > 0 \ \exists \delta = \frac{1}{2} \ \forall p, q \in X$

$d(p,q) < \frac{1}{2} \Rightarrow p = q$

$\Rightarrow d_Y(f(p), f(q)) = 0 < \varepsilon.$

So every function on (X, d_0) is uniformly continuous.

<u>Thm 4.20</u> HW to read.

Let E be a non-compact set in $\mathbb{R}^k$.

E is not compact $\Rightarrow$ either E is not closed
or E is not bounded.

(a1) E is not closed $\Rightarrow \exists\, p \in E' - E \neq \emptyset$.

$$f(x) = \frac{1}{\|x-p\|} : E \to \mathbb{R}.$$

$\forall n \in \mathbb{N}$ $\quad \underbrace{N_{\frac{1}{n}}(p) \cap E}$ $\neq \emptyset$.

$\forall x \in N_{\frac{1}{n}}(p) \cap E$, $\quad f(x) > n$

f is unbounded.

(a2) E is unbounded

$g(x) = d(p_0, x)$ for any fixed $p_0 \in X$

$g(x)$ unbounded.

(b1) $p \in E' - E \neq \emptyset$ $\qquad h(x) = \dfrac{1}{1 + d(x,p)} < 1$

no max, as $x \to p$

$h(x) \to 1$.

(b2) E is unbounded

$$\ell(x) = \frac{d(x, p_0)}{1 + d(x, p_0)} \to 1$$

$\ell(x)$ has no max. $\qquad$ as $d(x, p_0) \to \infty$.