Chap IV

CAUTION: Let $f:(X,d_X)\to(Y,d_Y)$ be continuous
$A\subseteq X$, $B\subseteq Y$.

A open $\not\Rightarrow$ $f(A)$ open (1)

A closed $\not\Rightarrow$ $f(A)$ closed (3)(a)

Thm A compact $\Rightarrow$ $f(A)$ compact (to prove today)

A bounded $\not\Rightarrow$ $f(A)$ bounded (2)

Prop B open $\Rightarrow$ $f^{-1}(B)$ open

Prop B closed $\Rightarrow$ $f^{-1}(B)$ closed

B compact $\not\Rightarrow$ $f^{-1}(B)$ compact (1)

B bounded $\not\Rightarrow$ $f^{-1}(B)$ bounded (1)

Ex (1) $f(x)\equiv 1:\mathbb{R}\to\mathbb{R}$ standard metric
$\mathbb{R}$ open, $f(\mathbb{R})=\{1\}$ not open
$B=\{1\}$ bounded $f^{-1}(\{1\})=\mathbb{R}$ unbounded
$B=\{1\}$ compact $f^{-1}(\{1\})=\mathbb{R}$ not compact

Example 2 $f:(0,1)\to(1,\infty)$

$f(x)=\frac{1}{x}$, $(0,1)$ bounded

$f((0,1))$ unbounded.

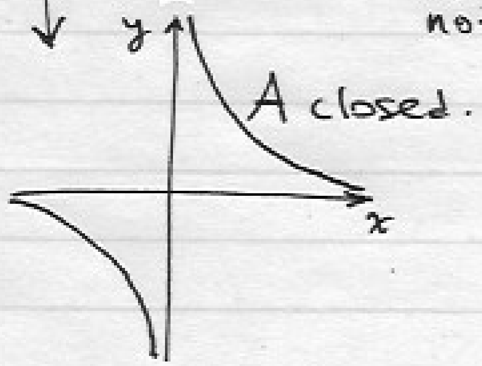
Example 3 $g(x,y)=x : \mathbb{R}^2\to\mathbb{R}$, continuous ("projection")

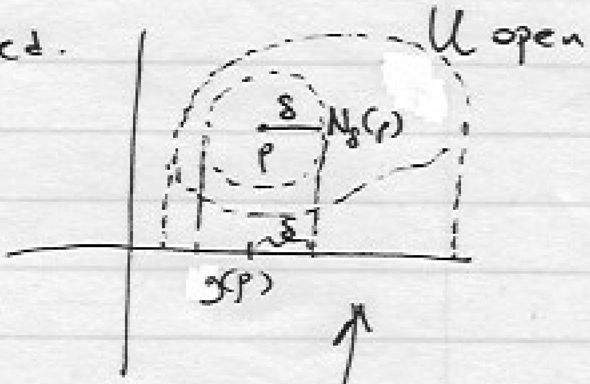
(a) $A=\{(x,y) \mid xy=1\}$ is a closed set

since $h(x,y)=xy : \mathbb{R}^2\to\mathbb{R}$

is continuous, $\underset{\text{closed}}{A} = h^{-1}(\underset{\text{closed.}}{\{1\}})$. $\Leftarrow$

$g(A)=\underbrace{\mathbb{R}-\{0\}}_{\text{not closed.}}$





(b) Although in example 1 we saw that f(open set) need not be an open set for arbitrary continuous functions; some functions such as $g(x,y)=x : \mathbb{R}^2\to\mathbb{R}$ satisfy "g(open set) is open". Called open maps. g is an open map.

This is because it takes interior points to interior points.

$$g(\underbrace{N_\delta(p)}_{\text{open in }\mathbb{R}^2}) = \underbrace{(g(p)-\delta,\ g(p)+\delta)}_{\text{open in }\mathbb{R}^1}$$

Theorem 4.14 *****

Let $f:(X,d_X)\to(Y,d_Y)$ be continuous, where (X,d_X) and (Y,d_Y) are metric spaces.

If K is a compact subset of X, then $f(K)$ is a compact subset of Y.

Proof Let $\{V_\alpha \mid \alpha\in\Lambda\}$ be an arbitrary open cover of $f(K)$, in Y

$$f(K)\subseteq\bigcup_{\alpha\in\Lambda}V_\alpha.\subseteq Y$$

By Thm 4.8:

$\forall\alpha$ V_α open, f continuous $f^{-1}(V_\alpha)$ is open in X.

$\forall\alpha$ Let $U_\alpha=f^{-1}(V_\alpha)$. U_α is open in X.

$\forall x\in K$, $f(x)\in f(K)\subseteq\bigcup_{\alpha\in\Lambda}V_\alpha$

$\exists\alpha_0$ s.t. $f(x)\in V_{\alpha_0}$.

$$x\in f^{-1}(V_{\alpha_0})=U_{\alpha_0}.\subseteq\bigcup_{\alpha\in\Lambda}U_\alpha$$

$\forall x\in K$, $x\in\bigcup_{\alpha\in\Lambda}U_\alpha$..

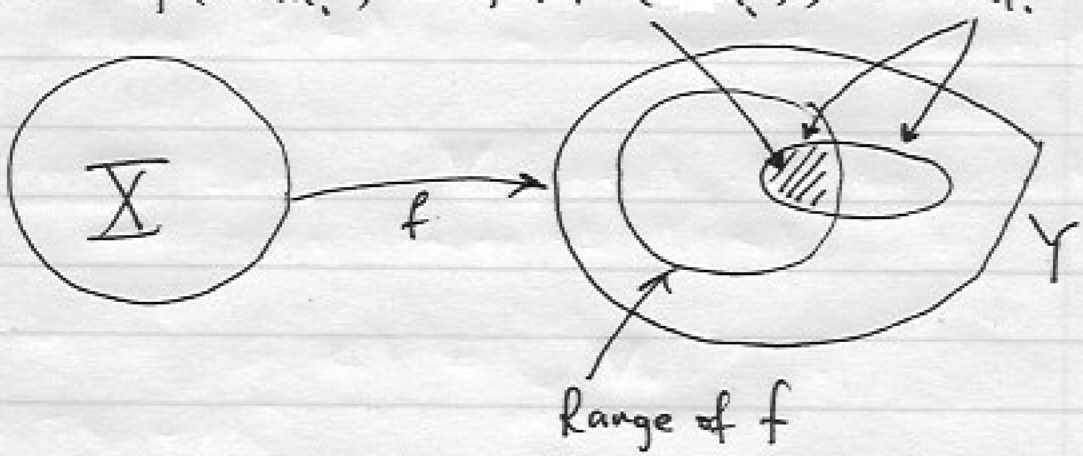
$K\subseteq\bigcup_{\alpha\in\Lambda}U_\alpha$, hence $\{U_\alpha\mid\alpha\in\Lambda\}$ is an open cover of K.

K is compact. $\exists\, \alpha_1, \alpha_2 \ldots \alpha_\ell$ s.t.

$$K \subseteq \bigcup_{i=1}^{\ell} U_{\alpha_i}$$

$$f(K) \subseteq f\Big(\bigcup_{i=1}^{\ell} U_{\alpha_i}\Big) = \bigcup_{i=1}^{\ell} f(U_{\alpha_i})$$

$$\forall i \quad f(U_{\alpha_i}) = f(f^{-1}(V_{\alpha_i})) \subseteq V_{\alpha_i}$$



$$f(K) \subseteq \bigcup_{i=1}^{\ell} V_{\alpha_i}$$

We showed that for every open cover $\{V_\alpha \mid \alpha \in \Lambda\}$ of $f(K)$, there exists a finite open subcover:

$$f(K) \subseteq \bigcup_{i=1}^{\ell} V_{\alpha_i}$$

$f(K)$ is compact.

<u>Def'n</u> A function $f:(X,d_X)\to(Y,d_Y)$ is called bounded if $f(X)$ is a bounded subset of Y: $\exists y_0\in Y, \exists R_0\in\mathbb{R}$ s.t.

$$f(X)\subseteq N_{R_0}(y_0).$$

(4.15) <u>Prop:</u> If $f:(X,d_X)\to(\mathbb{R}^k,\|\cdot\|)$ continuous, and X is compact, then $f(X)$ is compact, and hence closed & bounded. f is a bounded function.

<u>Theorem:</u> (Extreme Value Thm.) Let $X\neq\emptyset$, Let $f:(X,d)\to\mathbb{R}$, be continuous, and X be compact. Then $\exists\ p,q\in X$ s.t.

$\forall x\in X$

$$-\infty<\min_X f=f(p)\le f(x)\le f(q)=\max_X f<\infty.$$

4.15 is a corollary of 4.14.

Proof of 4.16.

$\mathbb{X}$ compact, f continuous $\Rightarrow f(\mathbb{X})$ compact in $\mathbb{R}$

$\Rightarrow f(\mathbb{X})$ is a bounded set in $\mathbb{R}$.

$\Rightarrow \exists m, M \in \mathbb{R}$ s.t.

$$-\infty < m = \inf(f(\mathbb{X})) \leq \sup(f(\mathbb{X})) = M < \infty$$

$f(\mathbb{X})$ closed since it is compact.

$$\inf(f(\mathbb{X})) = m \in \overline{f(\mathbb{X})} = f(\mathbb{X}) \quad \text{prop 2.28}$$
$$\sup(f(\mathbb{X})) = M \in \overline{f(\mathbb{X})} = f(\mathbb{X})$$

$$m \in f(\mathbb{X}) \Rightarrow \exists p \in \mathbb{X},\ f(p) = m$$
$$M \in f(\mathbb{X}) \Rightarrow \exists q \in \mathbb{X},\ f(q) = M.$$

$$\forall x \in \mathbb{X} \quad f(p) = m \leq f(x) \leq M = f(q).$$

See Thms 4.4 4.6

Thm 4.9 Let $f, g : (X, d_X) \to \mathbb{R}$ (or $\mathbb{C}$).

If f and g are continuous on X, then

$f \pm g$, $f \cdot g$, f/g $(g \neq 0)$ are continuous on X

Thm 4.10 Let $F = (\underbrace{f_1, f_2, \dots f_k}_{\text{components}}) : (X, d_X) \to (\mathbb{R}^k, \|\cdot\|)$.

F is continuous on $X \iff \forall j = 1, \dots k$, f_j is continuous on X.

Proof $\forall \vec{x} = (x_1, x_2, \dots x_k) \in \mathbb{R}^k$

$\forall j \quad |x_j| \leq \|\vec{x}\| = \sqrt{x_1^2 + \dots + x_k^2} \leq |x_1| + |x_2| + \dots + |x_k|$

Consequently $\forall p, q \in X$, $F(p), F(q) \in \mathbb{R}^k$, and

$$\forall j \quad |f_j(p) - f_j(q)| \leq \|F(p) - F(q)\| \leq \sum_{i=1}^{k} |f_i(p) - f_i(q)|$$

($\Rightarrow$:) Assume F is continuous.

For each j $\forall p \in X$ $\forall \varepsilon > 0$ $\exists \delta > 0$ $\forall q \in X$

$d(p,q) < \delta \Rightarrow |f_j(p) - f_j(q)| \leq \|F(p) - F(q)\| < \varepsilon$

($\Leftarrow$:) Assume each f_j is continuous.

$\forall p \in X$ $\forall \varepsilon > 0$ $\forall j$ $\exists \delta_j > 0$ $\forall q \in X$

$d_X(p,q) < \delta_j \Rightarrow |f_j(p) - f_j(q)| < \frac{\varepsilon}{k}$

Let $\delta = \min_i \delta_i > 0$.

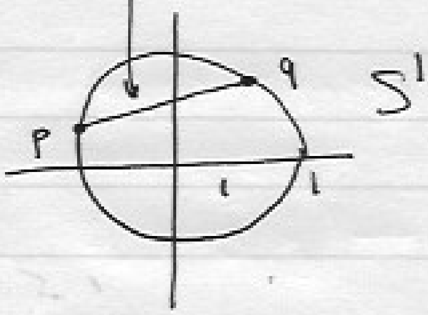
$\forall q \in X \quad d_X(p,q) < \delta \Rightarrow \|F(p) - F(q)\| \leq \sum_{j=1}^{k} |f_j(p) - f_j(q)|$

$< k \cdot \varepsilon/k = \varepsilon$.

Ex Let $f: \underbrace{\overset{\subseteq \mathbb{R}}{[0, 2\pi)}}_{|a-b| = dist(a,b)} \longrightarrow \underbrace{S^1 = \{(x,y) \mid x^2+y^2 = 1\}}_{d(p,q) = \|p-q\|} \subseteq \mathbb{R}^2$

be defined by

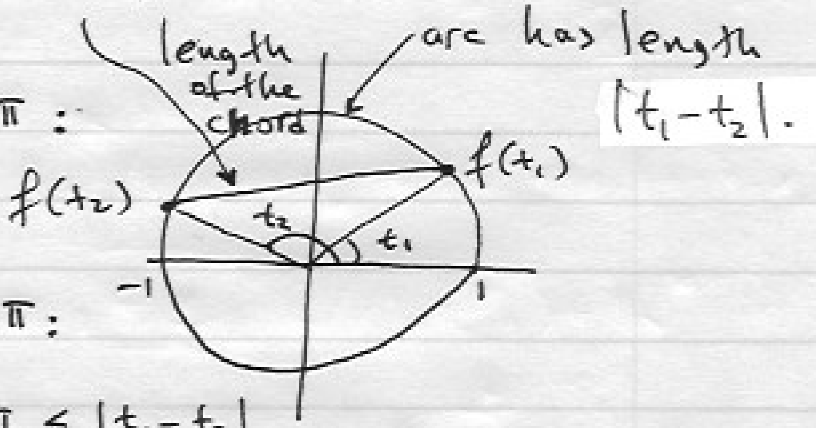
$f(t) = (\cos t, \sin t)$



f is continuous:

since $|f(t_1) - f(t_2)| \leq |t_1 - t_2|$:

Case 1 $|t_1 - t_2| \leq \pi$:



Case 2 $|t_1 - t_2| \geq \pi$:

$|f(t_1) - f(t_2)| \leq 2 \leq \pi \leq |t_1 - t_2|$

No proof:
- f is 1-1
- f is onto S^1

$\exists\, f^{-1} = g : S^1 \longrightarrow X = [0, 2\pi)$

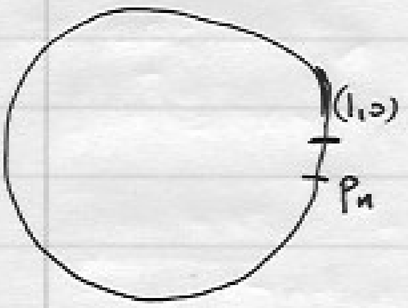
$(\cos t, \sin t) \longrightarrow t$

Claim g is not continuous (at $(1,0)$)

$g : S^1 \longrightarrow [0, 2\pi)$

g is not continuous at $(1,0)$:

METHOD I



Let $p_n = f\left(2\pi - \frac{1}{n}\right)$ $\forall n \in \mathbb{N}$

$$p_n = \left(\cos\left(2\pi - \tfrac{1}{n}\right), \sin\left(2\pi - \tfrac{1}{n}\right)\right)$$

$$\lim_{n\to\infty} p_n = (1,0) \text{ in } S^1$$

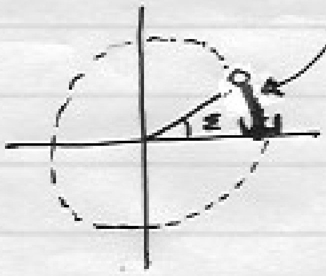
$$\lim_{n\to\infty} g(p_n) = \underbrace{\lim_{n\to\infty} \left(2\pi - \frac{1}{n}\right)}_{\text{actually DNE in } [0,2\pi)} = 2\pi \text{ in } \mathbb{R}$$

$$g\left(\lim_{n\to\infty} p_n\right) = g((1,0)) = 0$$

$\lim_{n\to\infty} g(p_n) \neq g\left(\lim_{n\to\infty} p_n\right) \Rightarrow g$ is not continuous at $(1,0)$

METHOD II $\forall \varepsilon > 0$ $\underbrace{[0, \varepsilon)}_{N_\varepsilon(0) \text{ in } [0,2\pi)}$ is open in $[0, 2\pi)$.

$g^{-1}([0,\varepsilon)) = f([0,\varepsilon))$ not open in S^1.



Use Thm 4.8.