CHAP IV

Defn Let (X, d_X), (Y, d_Y) be metric spaces,

$f: E \subseteq X \to Y$, $p \in E'$

We write

$$\lim_{x \to p} f(x) = q \quad \text{or} \quad f(x) \to q \text{ as } x \to p$$

for some $q \in Y$, if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t.

$$\forall x \in E,\ 0 < d_X(x,p) < \delta \Rightarrow d_Y(f(x), q) < \varepsilon.$$

Thm: $\lim_{x \to p} f(x) = q \iff$

$\forall \{p_n\}$ in E, $p_n \to p$, $\forall n$ $p_n \neq p$, one has $\lim_{n \to \infty} f(p_n) = q$

Proof: ($\Rightarrow$:) Given $\lim_{x \to p} f(x) = q$, ①

Given $\{p_n\}$ in E, $p_n \to p$, ($p_n \neq p$ $\forall n$), ②

To show $\lim_{n \to \infty} f(p_n) = q$.

(① $\Rightarrow$) $\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in E$

$$\left(0 < d_X(x, p) < \delta \Rightarrow d_Y(f(x), q) < \varepsilon \right)$$

(② $\Rightarrow$) $p_n \xrightarrow{\neq} p$, For the δ above $\exists N \in \mathbb{N}$

$$\forall n \geq N \quad \left(0 < d_X(p, p_n) < \delta \right)$$

and hence $d_Y(f(p_n), q) < \varepsilon$ by ①

Summarizing: $\forall \varepsilon > 0 \, \exists N \, \forall n \geq N$

$$d(f(p_n), q) < \varepsilon.$$

$\lim\limits_{n \to \infty} f(p_n) = q$, as a sequence.

($\Leftarrow$:) We will state the contrapositive:

$$\text{not}\left(\lim_{x \to p} f(x) = q \right) \Rightarrow \left\{ \text{not}\left(\forall \{p_n\}, \, p_n \xrightarrow{\neq} p, \, \lim_{n \to \infty} f(p_n) = q \right) \right.$$

(*) (**)

start with (*):

$$\text{not}\left(\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in E \left(0 < d_X(p, x) < \delta \Rightarrow d_Y(f(x), q) < \varepsilon \right) \right)$$

$$\exists \varepsilon > 0 \, \forall \delta > 0 \, \exists x_\delta \in E \left(0 < d_X(p, x_\delta) < \delta, \text{ but } d_Y(f(x_\delta), q) \geq \varepsilon \right)$$

We take $\delta = \frac{1}{n}$ $\forall n \in \mathbb{N}$

$$\exists \varepsilon > 0 \ \forall n \in \mathbb{N} \ \exists x_n \in E \ \Big(0 < d_X(p, x_n) < \tfrac{1}{n}, \text{ but } d_Y(f(x_n), q) \geq \varepsilon \Big)$$

We constructed a sequence $\{x_n\}_{n=1}^{\infty}$ in E s.t.

$x_n \to p$

$x_n \neq p$ $\forall n$

but $f(x_n) \not\to q$

$\underbrace{\lim f(x_n)}_{\text{may or may not exist.}} \neq q$

This is (**). Proved the contrapositive. #

<u>Corollary</u>: $\lim\limits_{x \to p} f(x) = q$ & $\lim\limits_{x \to p} f(x) = q'$

$\Rightarrow q = q'$. (via Thm 3.2.b p48)

<u>CONTINUOUS FUNCTIONS</u>

<u>Defn</u> Let (X, d_X), (Y, d_Y) be metric spaces

Let $f: E \subseteq X \to Y$, and $p \in E$.

- f is called continuous at p if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall x \in E$ $\big(d_X(p, x) < \delta \Rightarrow d_Y(f(p), f(x)) < \varepsilon. \big)$
- f is called continuous on E if f is continuous at each $p \in E$.

Consequently:

1. If $p \in E'$ as well as $p \in E$, then

$$f \text{ is continuous at } p \iff \lim_{x \to p} f(x) = f(p)$$

2. If $p \in E$, but $p \notin E'$, then f is automatically continuous at p.

Reason:

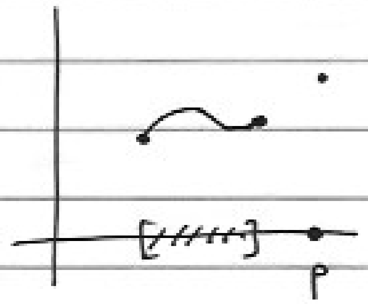
If $p \notin E'$, then $\exists\, \delta > 0$ s.t.

$$\phi = N_\delta(p) \cap (E - \{p\})$$

hence $N_\delta(p) \cap E = \{p\}$

$$x \in E, \quad d_X(x,p) < \delta \Rightarrow x = p.$$

$$\Rightarrow d_Y(f(x), f(p)) = 0 < \varepsilon.$$



<u>Example</u> Every function

$f : \mathbb{N} \longrightarrow (Y, d_Y)$ is continuous for every metric space. (Y, d_Y)

$\mathbb{N}$: standard metric $d(m,n) = |m-n|$

Prop Let $\mathbb{X}, Y, Z$ be metric spaces.

Let
$$\begin{array}{ccccc} E & \xrightarrow{f} & F & \xrightarrow{g} & Z, \\ \cap\!\shortmid & & \cap\!\shortmid & & \\ \mathbb{X} & & Y & & \end{array}$$

$f(E) \subseteq F$ so that $h(x) = g(f(x)) = g \circ f(x)$ is defined.

If f is continuous at p, and
g is continuous at $f(p)$,
then $h = g \circ f$ is continuous at p.

Proof: Let $\varepsilon > 0$ be given.
By continuity of g at $f(p)$, $\exists\, \eta > 0$ s.t.

(*) $\forall y \in F \left(d_Y(y, f(p)) < \eta \Rightarrow d_Z(g(y), g(f(p)) < \varepsilon \right)$

For $\eta > 0$ above, $\exists\, \delta > 0$ by the continuity of f at p

$$\forall x \in E \quad d_{\mathbb{X}}(x, p) < \delta \Rightarrow d_Y(f(x), f(p)) < \eta$$
$$\underset{\text{by (*)}}{\Longrightarrow} d_Z(g(f(x)), g(f(p)) < \varepsilon.$$

Hence $\forall \varepsilon > 0\ \exists\, \delta > 0$ s.t.

$$\forall x \in E,\ d_{\mathbb{X}}(x, p) < \delta \Rightarrow d_Z(g(f(x)), g(f(p)) < \varepsilon.$$

#.

<u>Thm 4.8</u> Let $f:(X, d_X) \longrightarrow (Y, d_Y)$.

f is continuous on X (as defined for metric spaces)

$\iff \forall U$ open set in Y, $f^{-1}(U)$ is open in X

<u>Remark:</u> This is the definition continuity in topological spaces.

<u>Recall</u> $\forall A \subseteq X$, $f(A) = \{f(x) \mid x \in A\} \subseteq Y$

$\forall B \subseteq Y$, $f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subseteq X$

<u>Proof of Thm</u>

($\Rightarrow$:) Assume f is continuous in the metric sense:

$$\circledast\ \forall p \in X\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in X$$
$$(d_X(x,p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon)$$

Let U be an open set in Y

(Want $f^{-1}(U)$ is an open set in X)

Let $p \in f^{-1}(U)$ be any point.

(want $\exists \delta > 0$ s.t. $N_\delta(p) \subseteq f^{-1}(U)$)

$p \in f^{-1}(U) \Longrightarrow f(p) \in U$, which is open.

$\exists \varepsilon > 0$ s.t. $N_\varepsilon(f(p)) \subseteq U$.

By $(*)$ $\exists\, \delta > 0$ s.t. $\forall x \in X$

$$d_X(x,p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon$$

$$\Rightarrow f(x) \in N_\varepsilon(f(p)) \subseteq U.$$

$$\Rightarrow f(x) \in U.$$

$$\Rightarrow x \in f^{-1}(U)$$

$$\forall x \quad x \in N_\delta(p) \Rightarrow x \in f^{-1}(U)$$

$N_\delta(p) \subseteq f^{-1}(U)$ for some $\delta > 0$.

p is an interior pt of $f^{-1}(U)$

p was chosen arbitrarily in $f^{-1}(U)$.

$f^{-1}(U)$ is an open set in X. $(\Rightarrow \checkmark)$

$(\Leftarrow:)$

Assume $\forall U$ open in Y, $f^{-1}(U)$ is open in X.

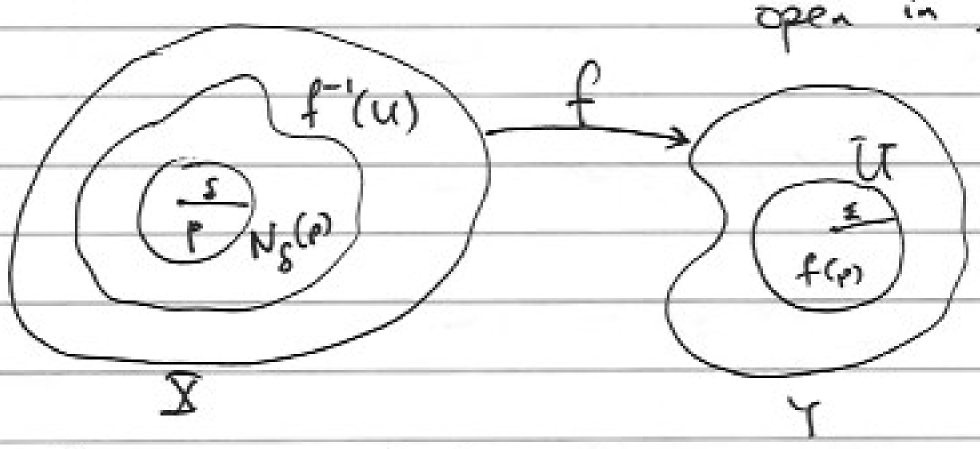
Want to show:

$$\left[\forall p \in X\ \forall \varepsilon > 0\ \exists\, \delta > 0\ \forall x \in X \quad (d_X(x,p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon\right.$$

Let $p \in X$, and $\varepsilon > 0$ be given.

Let $U = N_\varepsilon(f(p))$ open in Y.

Hypothesis $\Rightarrow$ $f^{-1}(U) = f^{-1}(N_\varepsilon(f(p)))$ is open in X.



$f(p) \in U = N_\varepsilon(f(p))$

$p \in f^{-1}(U)$ by def.

$f^{-1}(U)$ open $\Rightarrow \exists\, \delta > 0 \;\; N_\delta(p) \subseteq f^{-1}(U)$.

$$
\begin{aligned}
\forall x \in X \text{ s.t. } d(x,p) < \delta \;&\Rightarrow\; x \in N_\delta(p) \\
&\Rightarrow\; x \in f^{-1}(U) \\
&\Rightarrow\; f(x) \in U = N_\varepsilon(f(p)) \\
&\Rightarrow\; d_Y(f(x), f(p)) < \varepsilon.
\end{aligned}
$$

Hence we established

$\forall p \in X \; \forall \varepsilon > 0 \; \exists\, \delta > 0 \; \forall x \in X \; (d(x,p) < \delta \Rightarrow d(f(x), f(p)) < \varepsilon$

Corollary Let $f : (X, d_X) \to (Y, d_Y)$

f is continuous

$\iff \forall C$ closed in Y, $f^{-1}(C)$ is closed in X.

<u>Basic Lemma:</u> Let $f: X \to Y$ be a function.

(i) $\forall C \subseteq Y$, $X - f^{-1}(C) = f^{-1}(Y - C)$
(ii) $\forall C \subseteq Y$, $f(f^{-1}(C)) \subseteq C$

(iii) $\forall A, B \subseteq X$ $f(A \cup B) = f(A) \cup f(B)$
(iv) $\forall A, B \subseteq X$ $f(A \cap B) \subseteq f(A) \cap f(B)$

<u>Proof of (i)</u> Recall $x \in f^{-1}(C) \underset{①}{\iff} f(x) \in C$ (def)

$$\forall x \quad x \in X - f^{-1}(C) \iff x \notin f^{-1}(C)$$
$$\iff f(x) \notin C \quad \text{by ①}$$
$$\iff f(x) \in Y - C$$
$$\iff x \in f^{-1}(Y - C).$$

<u>Proof of Corollary:</u> $f: (X, d_X) \to (Y, d_Y)$

f is continuous on X
$\iff$ $\forall U$ open in Y, $f^{-1}(U)$ open in X
$\iff$ $\forall U$ open in Y, $X - f^{-1}(U)$ closed in X
$\iff$ $\forall U$ open in Y, $f^{-1}(Y - U)$ closed in X
$\iff$ $\forall\, Y - U$ closed in Y, $f^{-1}(Y - U)$ closed in X
$\iff$ $\forall\, C$ closed in Y, $f^{-1}(C)$ closed in X

via $Y - U = C$ substitution.

& Also Recall prop. 2.23.