ADDITION & MULTIPLICATION of SERIES

$\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$ converge

$$\Rightarrow \sum_{n=0}^{\infty} ca_n + db_n \text{ converges}$$
for given $c, d \in \mathbb{R}$.

Proof:

$$\sum_{n=0}^{k} ca_n + db_n = c\sum_{n=0}^{k} a_n + d\sum_{n=0}^{k} b_n$$

$$c \cdot \sum_{n=0}^{\infty} a_n + d \cdot \sum_{n=0}^{\infty} b_n$$

How do we multiply series?

In $\sum_{n=0}^{\infty} a_n b_n$, you are actually multiplying sequences

$a_0 \quad a_1 \quad a_2 \quad a_3 \quad \dots$
$b_0 \quad b_1 \quad b_2 \quad b_3 \quad \dots$

Multiply the sums $\begin{cases} (a_0 + a_1 + a_2 + a_3 + a_4) \\ (b_0 + b_1 + b_2 + b_3 + b_4) \end{cases}$

$$= a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) +$$
$$+ (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0) + \text{etc.}$$

Defn Given $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$

define $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

$\sum_{n=0}^{\infty} c_n$ is called the Cauchy product of

$\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$.

Example ①

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots = \frac{2}{3}$$

$c_0 = 1$

$c_1 = \frac{-1}{2} + \frac{1}{2} = 0$ $\qquad c_{2n} = \left(\frac{1}{4}\right)^n$

$c_2 = \frac{1}{4} - \frac{1}{4} + \frac{1}{4} = \frac{1}{4}$ $\qquad c_{2n+1} = 0$

$c_3 = -\frac{1}{8} + \frac{1}{8} - \frac{1}{8} + \frac{1}{8} = 0$

$$\sum_{n=0}^{\infty} c_n = 1 + \frac{1}{4} + \frac{1}{16} + \cdots = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}.$$

$$\underbrace{\left(\sum_{n=0}^{\infty} \frac{1}{2^n}\right)}_{2} \underbrace{\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}\right)}_{\frac{2}{3}} = \underbrace{\left(\sum_{n=0}^{\infty} c_n\right)}_{\frac{4}{3}}$$

True for this case.

Theorem 3.50 (Mertens)

If (1) $\sum_{n=0}^{\infty} a_n$ converges absolutely, and

(2) $\sum_{n=0}^{\infty} a_n = A$

(3) $\sum_{n=0}^{\infty} b_n = B$ } $A, B \in \mathbb{R}$, and

(4) $c_n = \sum_{k=0}^{n} a_k b_{n-k}$,

then $\sum_{n=0}^{\infty} c_n = AB$

Proof Let $A_n = \sum_{k=0}^{n} a_k$

$$B_n = \sum_{k=0}^{n} b_k$$

$$\beta_n = B_n - B$$

$$C_n = \sum_{k=0}^{n} c_k.$$

We know $A_n \to A$, $B_n \to B$, $\beta_n \to 0$,
We want $C_n \to AB$.

$$C_n = a_0 b_0 + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0).$$

$$= a_0 B_n + a_1 B_{n-1} + a_2 B_{n-2} + \cdots + a_n B_0$$

$$C_n = a_0(B+\beta_n) + a_1(B+\beta_{n-1}) + \cdots + a_n(B+\beta_0)$$

$$= B\underbrace{(a_0 + a_1 + \cdots a_n)}_{A_n} + \underbrace{a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0}_{\text{define } \gamma_n}$$

$$C_n = BA_n + \gamma_n.$$

know $BA_n \xrightarrow{\text{as } n\to\infty} BA$, want to show $\gamma_n \to 0$

$\sum_{n=0}^{\infty} a_n$ converges absolutely,

$\exists \alpha \in \mathbb{R}$ s.t. $\alpha = \sum_{n=0}^{\infty} |a_n|$. $(*)$

Let $\varepsilon > 0$ be given

$\exists N \in \mathbb{N}, \forall n \geq N \; |\beta_n| \leq \varepsilon$ (since $\beta_n \to 0$) $(**)$

$\forall n \geq N$:

$$|\gamma_n| = |a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0|$$

$$\leq \underbrace{|\beta_0 a_n + \beta_1 a_{n-1} + \cdots + \beta_N a_{n-N}|}_{\text{(i) want } \to 0 \text{ as } n\to\infty} + \underbrace{|\beta_{N+1}a_{n-(N+1)} + \cdots + \beta_n a_0|}_{\text{want } \leq \alpha\varepsilon \text{ (ii)}}$$

$\forall n \geq N$

(i) $|\beta_0 a_n + \beta_1 a_{n-1} + \cdots + \beta_N a_{n-N}| \to 0,$ as $n \to \infty$

since N fixed, $\beta_0, \beta_1, \ldots \beta_N$ fixed

$N \leq n \to \infty$

$a_n \to 0 \quad (\Leftarrow \sum_{n=0}^{\infty} a_n$ converges$)$

(ii) $|\beta_{N+1} a_{n-(N+1)} + \cdots + \beta_n a_0|$

$$= \left| \sum_{k=N+1}^{n} \beta_k a_{n-k} \right| \leq \sum_{k=N+1}^{n} |\beta_k| |a_{n-k}|$$

$$\leq \varepsilon \sum_{k=N+1}^{n} |a_{n-k}| \leq \varepsilon \sum_{k=0}^{\infty} |a_k| = \varepsilon \alpha.$$

(**) p 4

$$|\gamma_n| = \left| \sum_{k=0}^{n} \beta_k a_{n-k} \right| \leq \left| \sum_{k=0}^{N} \beta_k a_{n-k} \right| + \left| \sum_{k=N+1}^{n} \beta_k a_{n-k} \right|$$

$$\leq \underbrace{\left| \sum_{k=0}^{N} \beta_k a_{n-k} \right|}_{\downarrow^{(i)}\ 0} + \underbrace{\varepsilon \alpha.}_{(ii)}$$

$(N$ fixed, ε fixed, $n \to \infty)$

$\Rightarrow \limsup_n |\gamma_n| \leq \varepsilon \alpha \quad \forall \varepsilon > 0$

$\Rightarrow \limsup_n |\gamma_n| = 0$

$\Rightarrow \lim_n \gamma_n = 0$

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty} (B A_n + \gamma_n) = BA. \qquad \#$$

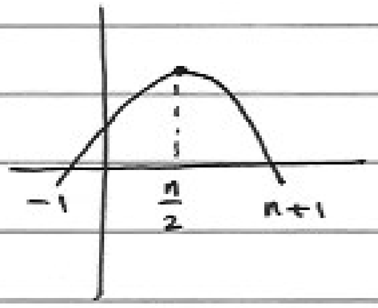
Example 2 $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^p}.$

$\sum a_n$ converges absolutely if $p > 1$

converges conditionally if $0 < p \leq 1$.

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} \frac{(-1)^k}{(k+1)^p} \cdot \frac{(-1)^{n-k}}{(n-k+1)^p}$$

$$= (-1)^n \sum_{k=0}^{n} \frac{1}{(k+1)^p (n-k+1)^p}$$



$f(x) = -(x+1)(x-(n+1)) \leq f(\frac{n}{2})$

$\forall k = 0, 1, 2, \ldots n$

$(k+1)(n+1-k) \leq \left(1 + \frac{n}{2}\right)^2 = \left(\frac{2+n}{2}\right)^2$

$$\left(\frac{1}{(k+1)(n-k+1)}\right)^p \geq \left(\frac{2}{2+n}\right)^{2p}$$

$$|c_n| = \sum_{k=0}^{n} \frac{1}{(k+1)^p (n-k+1)^p} \geq \underbrace{(n+1)\left(\frac{2}{n+2}\right)^{2p}}$$

converges to 2 if $p = \frac{1}{2}$

converges to ∞ if $p < \frac{1}{2}$

If $0 < p \leq \frac{1}{2}$ then $\lim c_n \neq 0$

hence $\sum c_n$ is divergent

even though

$\sum a_n = \sum b_n$ is convergent

by Alternating Series Test.

REARRANGEMENTS (Shuffling the terms of a series)

Def: Let $\{a_n\}$ be a sequence, $f: \mathbb{N} \to \mathbb{N}$ be a bijection, $k_n = f(n)$.

$a_n' = a_{f(n)} = a_{k_n}$ is called a rearrangement of $\{a_n\}$

Thm: Let $\sum a_n'$ be a rearrangement of $\sum a_n$, and $\sum a_n$ be absolutely convergent.
$\Rightarrow$ $\sum a_n'$ is also convergent, $\sum a_n' = \sum a_n$

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2.$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \ln 2$$

Next, we will rearrange the terms of this series so that the new series will converge to $0 \neq \ln 2$.

Obs: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is not absolutely convergent.

$a_n = 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \frac{1}{7}, -\frac{1}{8}, \cdots$

<u>Want</u> a rearrangement a_n' s.t. $\sum a_n' = 0$.

$0 < 1$

$0 < 1 - \frac{1}{2}$

$0 < 1 - \frac{1}{2} - \frac{1}{4}$

$0 < 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6}$

$0 > 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} \approx -0.04167$

$0 < 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{3} \approx 0.29167$

$\vdots$

$0 < 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{3} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14}$

$0 > 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{3} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16}$

$0 < (\quad '' \quad) + \frac{1}{5} \approx 0.17440$

$0 < (\quad '' \quad) + \frac{1}{5} - \frac{1}{18} - \frac{1}{20} - \frac{1}{22}$

$0 > (\quad '' \quad) + \left(\frac{1}{5} - \frac{1}{18} - \frac{1}{20} - \frac{1}{22} - \frac{1}{24}\right)$

≈ -0.01827

$0 < (\quad + \quad) + \frac{1}{7} \approx 0.12459$

$(\quad '' \quad) + \frac{1}{7} - \frac{1}{26} - \frac{1}{28} - \frac{1}{30} - \frac{1}{32}$

$\approx -0.01417.$

Rearranging in the following order

1 2 4 6 8 3 10 12 14 16 5 18 20 22 24 7 26 28 30 32 9 ....

Why should this work? PTO

Theorem (Riemann)

Let $\sum a_n$ be a series which converges (conditionally) but not absolutely.

Let $-\infty \le \alpha \le \beta \le \infty$ be given

Then $\exists$ a rearrangement $\sum a_n'$ of $\sum a_n$ s.t.

$$s_n' = \sum_{k=0}^{n} a_k'$$

$$\liminf s_n' = \alpha$$
$$\limsup s_n' = \beta.$$

Corollary: (Take $\alpha = \beta$.)
Let $\sum a_n$ be convergent but not absolutely. Let $\alpha \in [-\infty, \infty]$. Then $\exists$ rearrangement $\sum a_n'$ of $\sum a_n$ s.t.

$$\sum_{n=0}^{\infty} a_n' = \alpha.$$

Proof p76-77, is quite similar to the example we have done above with $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, (but done rigorously).

HW to read.