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Absolute Convergence

Defn A series $\sum_{n=1}^{\infty} a_n$ is called to converge absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.

Thm If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges.

Converse is false $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges but $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

Proof: (CCPS for $\sum |a_n| \Rightarrow$ CCPS for $\sum a_n$)

$\forall \varepsilon > 0 \ \exists N \ \forall n, m \geq N \quad m \geq n \quad \left| \sum_{k=n}^{m} |a_k| \right| < \varepsilon.$

$$\left| \sum_{k=n}^{m} a_k \right| \leq \sum_{k=n}^{m} |a_k| = \left| \sum_{k=n}^{m} |a_k| \right| < \varepsilon.$$

Hence
$\forall \varepsilon > 0 \ \exists N \ \forall n, m \geq N, \quad m \geq n \quad \left| \sum_{k=n}^{m} a_k \right| < \varepsilon.$

Then Recall Thm 3.22 p 59.

Summation by Parts:

<u>Lemma:</u> Given $\{a_n\}$, $\{b_n\}$ sequences

Define

$$A_n = \sum_{k=0}^{n} a_k, \quad A_{-1} = 0$$

Then

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) - A_q b_q - A_{p-1} b_p$$

<u>Proof</u>

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} \underbrace{(A_n - A_{n-1})}_{a_n} b_n$$

$$= \sum_{n=p}^{q} A_n b_n - \sum_{n=p}^{q} A_{n-1} b_n$$

subst. n for $n-1$

$$= \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1}$$

$$= \underbrace{\sum_{n=p}^{q-1} (A_n b_n - A_n b_{n+1})}_{\text{common } n\text{'s in } \Sigma\text{'s}} + \underbrace{A_q b_q - A_{p-1} b_p}_{\text{remaining terms}}.$$

3.42

Thm: Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be given s.t.

(i) $A_n = \sum_{k=0}^{n} a_k$ is a bounded sequence

(ii) $b_0 \geq b_1 \geq b_2 \geq \dots b_n \geq b_{n+1}$, $b_n \downarrow$.

(iii) $\lim_{n\to\infty} b_n = 0$ $\quad$ (ii, iii $\Rightarrow b_n \geq 0 \ \forall n$)

Then $\sum_{n=0}^{\infty} a_n b_n$ converges

Proof Let $\varepsilon > 0$ be given.
$\{A_n\}$ is bounded. $\exists M \ \forall n \ |A_n| \leq M$.

$b_n \downarrow 0 \ \exists N \quad 0 \leq b_N < \frac{\varepsilon}{2M}$
$(\Rightarrow \forall n \geq N \quad 0 \leq b_n \leq b_N < \frac{\varepsilon}{2M})$

We want CCPS for $\sum_{n=1}^{\infty} a_n b_n$,
and use Thm 3.22

PTO

$\forall p, q \geq N, \ q \geq p:$

$$\left| \sum_{k=p}^{q} a_k b_k \right|$$

by Summation by Parts p 2

$$= \left| \sum_{k=p}^{q-1} A_k (b_k - b_{k+1}) - A_q b_q - A_{p-1} b_p \right|$$

(since $|A_n| \leq M$ and $b_n \geq 0$)

$$\leq M \left(\sum_{k=p}^{q-1} |b_k - b_{k+1}| + b_q + b_p \right)$$

(use $b_k \geq b_{k+1}$)

$$\leq M (b_p \underbrace{- b_{p+1} + b_{p+1}}_{0} \underbrace{- b_{p+2} + \cdots + b_{q-1}}_{0} \underbrace{- b_q + b_q}_{0} + b_p)$$

$$= 2 M b_p \leq 2 M b_N < \varepsilon.$$

CCPS for $\sum a_n b_n$ holds, hence it converges. Thm 3.22. #

Corollary Alternating Series Test

If $b_0 \geq b_1 \geq b_2 \geq \cdots \geq b_n \geq b_{n+1} \geq 0$ and $\lim_{n \to \infty} b_n = 0$,

then $\sum_{n=0}^{\infty} (-1)^n b_n$ converges

Proof of Alternating Series Test

Take $a_n = (-1)^n$

$a_n : 1, -1, 1, -1, 1, -1 \ldots$

$n \geq 0$

$A_n : 1, 0, 1, 0, 1, \ldots$ bounded.

$$A_n = \sum_{k=0}^{n} a_k$$

Hence $\sum_{n=0}^{\infty} a_n b_n = \sum_{n=0}^{\infty} (-1)^n b_n$ converges by Thm 3.42.

Examples

Recall $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\iff p > 1$.

Since (i) $\frac{1}{n^p} \downarrow 0$ for $p > 0$ & (ii) $\frac{1}{n^p} \geq 1$ for $p \leq 0$

$\sum \frac{(-1)^n}{n^p}$ converges $\iff p > 0$

such as $\sum_{n=1}^{\infty} \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverge

but $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converge.

Many examples in Calculus II.

Example Let $m \in \mathbb{N}$, $m \geq 2$ be fixed

$*$ For which p does $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(2\pi n/m)}{n^p}$ converge ?

$p > 1$, answer is easy: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges, so use comparison Thm.

Actually $(*)$ is convergent for $p > 0$.

Lemma: If $m \geq 2$, then

$$\sum_{n=0}^{m-1} \cos\left(\frac{2\pi n}{m}\right) = \sum_{n=1}^{m} \cos \frac{2\pi n}{m} = 0.$$

Proof of Lemma: $e^{i\theta} = \cos\theta + i \sin\theta$.

Let $\sigma = e^{2\pi i/m} = \cos\frac{2\pi}{m} + i \sin\frac{2\pi}{m}$

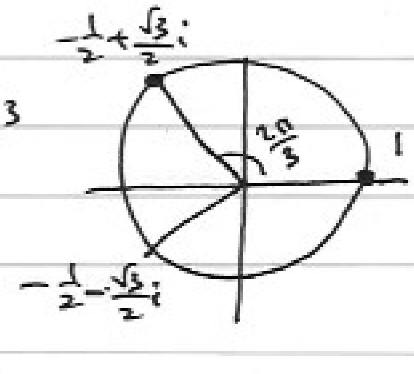
$$\sigma^n = e^{2\pi i n/m} = \cos\frac{2\pi n}{m} + i \sin\frac{2\pi n}{m}$$

$$\sigma^m = 1$$

$$0 = \sigma^m - 1 = (\underbrace{\sigma - 1}_{m \geq 2 \Rightarrow \neq 0})(\underbrace{\sigma^{m-1} + \sigma^{m-2} + \cdots + \sigma + 1}_{0})$$

$$\sum_{n=0}^{m-1} \sigma^n = 0 = \Big(\underbrace{\sum_{n=0}^{m-1} \cos\frac{2\pi n}{m}}_{0}\Big) + i\Big(\underbrace{\sum_{n=0}^{m-1} \sin\frac{2\pi n}{m}}_{0}\Big)$$

Ex $m = 3$



Let $a_n = \cos\frac{2\pi n}{m} = a_{n+mp}$. $\forall p \in \mathbb{N}$
periodic

$0 = a_0 + a_1 + \cdots + a_{m-1} = a_1 + a_2 + \cdots + a_m$
$= a_{m+1} + a_{m+2} + \cdots + a_{2m}$ etc.

$\forall n \in \mathbb{N}$, $n = mp + r$, $p \in \mathbb{N}\cup\{0\}$, $r \in \mathbb{N}\cup\{0\}$
$\uparrow$ remainder $0 \le r < m$

$$A_n = \sum_{k=1}^{n} a_k = \underbrace{\sum_{k=1}^{mp} a_k}_{0} + \sum_{k=mp+1}^{n} a_k = A_r$$

(Take $A_0 = 0$)

$\forall n \quad |A_n| \le M = \max(A_1, A_2, \ldots A_m) \le m$. actually

By Thm 3.42 $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{\cos(2\pi m/n)}{n^p}$

converges for $p > 0$ and $m \geq 2$,

since $\frac{1}{n^p} \to 0$, $\frac{1}{n^p} \geq \frac{1}{(n+1)^p} > 0$
$\forall p > 0$.

False for $m = 1$, $p \le 1$

$$\sum_{n=1}^{\infty} \frac{\cos(2n\pi/1)}{n^p} = \sum_{n=1}^{\infty} \frac{1}{n^p} = +\infty.$$

ADDITION & MULTIPLICATION of SERIES

$$\sum_{n=0}^{\infty} a_n, \sum_{n=0}^{\infty} b_n \text{ converge}$$

$$\Rightarrow \sum_{n=0}^{\infty} ca_n + db_n \text{ converges for given } c, d \in \mathbb{R}.$$

Proof:

$$\sum_{n=0}^{k} ca_n + db_n = c\underbrace{\sum_{n=0}^{k} a_n}_{\downarrow} + d\underbrace{\sum_{n=0}^{k} b_n}_{\downarrow}$$

$$c \cdot \sum_{n=0}^{\infty} a_n + d \cdot \sum_{n=0}^{\infty} b_n$$

How do we multiply series?

In $\sum_{n=0}^{\infty} a_n b_n$, you are actually multiplying sequences

$$\begin{matrix} a_0 & a_1 & a_2 & a_3 & \dots \\ b_0 & b_1 & b_2 & b_3 & \dots \end{matrix}$$

Multiply the sums $\begin{cases} (a_0 + a_1 + a_2 + a_3 + a_n) \\ (b_0 + b_1 + b_2 + b_3 + b_n) \end{cases}$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) +$$
$$+ (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0) + \text{ etc.}$$

Defn Given $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$

define $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

$\sum_{n=0}^{\infty} c_n$ is called the Cauchy product of

$\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$.

Example ①

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots = \frac{2}{3}$$

$c_0 = 1$

$c_1 = \frac{-1}{2} + \frac{1}{2} = 0$

$c_2 = \frac{1}{4} - \frac{1}{4} + \frac{1}{4} = \frac{1}{4}$

$c_3 = -\frac{1}{8} + \frac{1}{8} - \frac{1}{8} + \frac{1}{8} = 0$

$c_{2n} = \left(\frac{1}{4}\right)^n$

$c_{2n+1} = 0$

$$\sum_{n=0}^{\infty} c_n = 1 + \frac{1}{4} + \frac{1}{16} + \cdots = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}.$$

$$\underbrace{\left(\sum_{n=0}^{\infty} \frac{1}{2^n}\right)}_{2}\underbrace{\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}\right)}_{\frac{2}{3}} = \underbrace{\left(\sum_{n=0}^{\infty} c_n\right)}_{\frac{4}{3}}$$

True for this case.