Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be given, $a_n \neq 0$ $\forall n$.

Let $\alpha = \limsup_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

(a) If $\alpha < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\left| \frac{a_{n+1}}{a_n} \right| \geq 1$ $\forall n \in \mathbb{N}$, $n \geq N$ for some N
then $\sum a_n$ diverges

Proof of (b)

$0 \neq |a_n| \leq |a_{n+1}| \leq \ldots \leq |a_{n+k}|$ $\forall k \geq 0$

Fix N $\forall n \geq N$ $\quad |a_n| \geq |a_N| > 0$

$\lim_{n\to\infty} a_n \neq 0.$

(Recall that $\sum a_n$ convergent $\Rightarrow \lim a_n = 0$)

Hence $\sum a_n$ is not convergent.

Proof of Ratio test (a)

Since $0 \le \alpha < 1$

$\exists \beta \quad 0 \le \alpha < \beta < 1.$

$\exists N \quad \forall n \ge N$

$$\left| \frac{a_{n+1}}{a_n} \right| \le \beta \text{ since } \beta > \alpha$$

(recall Prop. 3.17b on March 13)

$$|a_{n+1}| \le \beta |a_n| \quad \forall n \ge N$$

$$|a_{N+1}| \le \beta |a_N|$$

$$|a_{N+2}| \le \beta |a_{N+1}| \le \beta^2 |a_N|$$

$$\vdots$$

$$|a_{N+k}| \le \beta^k |a_N|.$$

Use Comparison Test Thm 3.25

Compare $\sum_{k=0}^{\infty} a_{N+k}$ to $\underbrace{\sum_{k=0}^{\infty} \beta^k |a_N|}_{\text{convergent since } 0 \le \beta < 1}$ fixed #

$$\sum_{k=0}^{\infty} a_{N+k} = \sum_{n=N}^{\infty} a_n, \text{ it is convergent}$$

$$\sum_{n=0}^{\infty} a_n = \underbrace{a_0 + a_1 + \cdots + a_{N-1}}_{\text{finite sum.}} + \sum_{n=N}^{\infty} a_n \text{ is convergent.}$$

Ex Compare Root & Ratio tests on $\sum_{n=0}^{\infty} a_n$ where

$$a_n = \begin{cases} \left(\frac{1}{4}\right)^{n/2} & \text{if } n \text{ is even} \\ \left(\frac{1}{9}\right)^{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

$\limsup \sqrt[n]{|a_n|} = \frac{1}{2} < 1$, hence this series converges, by root test

Ratio test

$$\frac{a_{2n+1}}{a_{2n}} = \left(\frac{1}{9}\right)^n \Big/ \left(\frac{1}{4}\right)^n = \left(\frac{4}{9}\right)^n \longrightarrow 0$$

$$\frac{a_{2n}}{a_{2n-1}} = \left(\frac{1}{4}\right)^n \Big/ \left(\frac{1}{9}\right)^{\frac{2n-1-1}{2}} = \left(\frac{1}{4}\right)^n \Big/ \left(\frac{1}{9}\right)^{n-1}$$

$$= 9 \cdot \left(\frac{9}{4}\right)^n \longrightarrow \infty.$$

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = +\infty, \quad \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$$

Ratio test is inconclusive,

Also:

$$\underbrace{\liminf_n \frac{a_{n+1}}{a_n}}_{0} \le \underbrace{\liminf_n \sqrt[n]{|a_n|}}_{\frac{1}{3}} \le \underbrace{\limsup_n \sqrt[n]{|a_n|}}_{\frac{1}{2}} \le \underbrace{\limsup \left|\frac{a_{n+1}}{a_n}\right|}_{+\infty}$$

Thm Let $\{c_n\}$ be a sequence of positive real numbers. Then

$$\liminf_n \frac{c_{n+1}}{c_n} \le \underbrace{\liminf_n \sqrt[n]{c_n} \le \limsup \sqrt[n]{c_n}}_{\text{obvious}} \underset{①}{\le} \limsup_n \frac{c_{n+1}}{c_n}$$

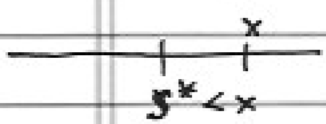
Proof of ①: WTS $\limsup_n \sqrt[n]{c_n} \le \limsup_n \frac{c_{n+1}}{c_n}$.

If $\limsup \frac{c_{n+1}}{c_n} = +\infty$, there is nothing to prove.

So we can assume $\limsup_n \frac{c_{n+1}}{c_n} = \alpha \in \mathbb{R}$.

Choose $\beta \in \mathbb{R}$ s.t. $\alpha < \beta < \infty$.

By using Prop 3.17 $\exists N \ \forall n \ge N \quad \frac{c_{n+1}}{c_n} \le \beta$.



$$c_{N+1} \le \beta c_N$$
$$c_{N+2} \le \beta c_{N+1} \le \beta^2 c_N$$
$$\vdots$$
$$c_{N+k} \le \beta^k c_N$$

substitute $m = N+k$

$\forall m \ge N \qquad c_m \le \beta^{m-N} c_N$

$$\sqrt[m]{c_m} \le \left(\frac{\beta^m}{\beta^N} c_N\right)^{\frac{1}{m}} = \beta \cdot \sqrt[m]{\frac{1}{\beta^N}} \sqrt[m]{c_N} \to \beta \cdot 1 \cdot 1$$

N fixed, $m \to \infty$

By Prop. 3.19 $\quad \limsup \sqrt[m]{c_m} \le \beta. \quad \forall \beta > \alpha.$

$$\limsup \sqrt[m]{c_m} \le \alpha. \qquad \#.$$

Power Series

$\sum_{n=0}^{\infty} c_n (x-a)^n$ in $\mathbb{R}$ centered at a.

Formal sum, convergency depends on the choice of x.

Ex $\sum_{n=0}^{\infty} x^n$ converges if $|x|<1$
diverges if $|x| \geq 1$

Convention: $n=0$, $x=0$, we take it as 1.

Actually we know if $|x|<1$, then

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots = \frac{1}{1-x}.$$

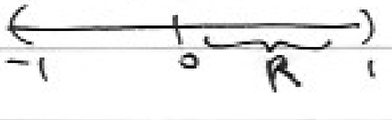
If $|x| \geq 1$, this equality is not true: in general

$$x=2 \qquad \underbrace{\sum_{n=0}^{\infty} x^n = 1+2+4+\cdots}_{\infty} \neq \frac{1}{1-2} = -1$$

$$x=1 \qquad \sum_{n=0}^{\infty} x^n = 1+1+1+1\cdots \quad \text{divergent}$$

$$x=-1 \qquad \sum_{n=1}^{\infty} (-1)^n = \underbrace{1-1+1-1+1\cdots}_{\neq \frac{1}{1-(-1)} = \frac{1}{2}} \quad \text{divergent}$$

Domain of convergence: $(-1,1)$



R = Radius of convergence is 1

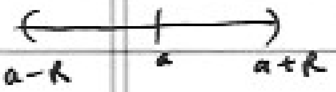
3.39 Theorem: Given $\sum_{n=0}^{\infty} c_n (x-a)^n$,

let $\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$, and let $R = \frac{1}{\alpha}$.

(if $\alpha = 0$, take $R = \infty$;
if $\alpha = \infty$, take $R = 0$)

Then

$\sum_{n=0}^{\infty} c_n (x-a)^n$ converges for $|x-a| < R$
diverges for $|x-a| > R$
inconclusive for $|x-a| = R$.



Proof Straightforward application of Root test.

Fix $x \in \mathbb{R}$, set $a_n = c_n (x-a)^n$

$$\sqrt[n]{|a_n|} = \sqrt[n]{|c_n||x-a|^n} = \sqrt[n]{|c_n|} \cdot |x-a|$$

$$\limsup_n \sqrt[n]{|a_n|} = \limsup_n |x-a| \cdot \sqrt[n]{|c_n|} = |x-a| \cdot \alpha$$

since $|x-a| \geq 0$
& fixed.

Root test convergent $\Leftarrow \begin{cases} \alpha|x-a| < 1 & \text{i.e.} \\ |x-a| < \frac{1}{\alpha} = R. \end{cases}$

divergent $\Leftarrow \begin{cases} \alpha|x-a| > 1 & \text{i.e} \\ |x-a| > \frac{1}{\alpha} = R. \end{cases}$

Examples

(a) $\sum_{n=0}^{\infty} x^n$, $c_n = 1$, $\limsup \sqrt[n]{c_n} = 1 = \alpha$

$R = \frac{1}{\alpha} = 1$

Domain of convergence $(-1, 1)$

Discussed divergence at ± 1

(b) $\sum_{n=0}^{\infty} (x-a)^n \frac{1}{n!} = e^{x-a}$

$c_n = \frac{1}{n!}$ $\sqrt[n]{c_n} = \left(\frac{1}{n!}\right)^{\frac{1}{n}} \to 0$ since

Stirlings formula $n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$

$\alpha = 0$, $R = \infty$, Domain of convergence $\mathbb{R}$.

(c) $\sum_{n=0}^{\infty} n! (x-a)^n$, $\alpha = \infty$

$R = 0$

Domain of convergence $\{a\}$.

More Examples

(d) $\sum_{n=0}^{\infty} n(x-a)^n$ $\quad \sqrt[n]{n} \to 1 = \alpha, \quad R = 1,$

converges on $(a-1, a+1)$

End pts. Take $a = 0$ for simplicity

$\sum_{n=0}^{\infty} n x^n$

diverges at both end pts.

$x = 1 \quad +1+2+3+4+\cdots \to \infty.$

$x = -1 \quad -1+2-3+4-5+6 \cdots$

partial sums $-1, 1, -2, 2, -3, 3$ diverges

(e) $\sum_{n=1}^{\infty} \frac{1}{n} x^n$ $\quad \alpha = 1, \ R = 1$

Converges on $[-1, 1)$

different behavior at each end pt.

$x = 1 \quad \sum \frac{1}{n} \cdot 1^n = +\infty.$

$x = -1 \quad \sum \frac{1}{n}(-1)^n = -1 + \frac{1}{2} - \frac{1}{3} \cdots\cdots = -\ln 2$

we will discuss alternating series later.

(f) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}(x-3)^n$ $\quad \sqrt[n]{\frac{2^n}{n^2}} \Rightarrow 2 = \alpha \quad R = \frac{1}{2}.$

domain of convergence

$\left[3 - \frac{1}{2}, 3 + \frac{1}{2}\right]$

Converges at both end pts.

$x = \frac{5}{2}$

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2}\left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} < \infty.$$

$\Uparrow$ (Thm 3.25)

$x = \frac{7}{2}$

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2}\left(+\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \quad \text{(Thm 3.28)}$$