March 13, 2020

§3 Series Continue

①

Ex

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \text{ converges} \iff p>1$$

We will use Thm 3.27 p61.

$$a_n = \frac{1}{n(\ln n)^p}$$

$$2^k a_{2^k} = \frac{2^k}{2^k(\ln 2^k)^p} = \frac{1}{(k\ln 2)^p} = \frac{1}{(\ln 2)^p}\cdot\frac{1}{k^p}$$

$$\sum \frac{1}{k^p} \text{ converges} \iff p>1.$$

$$\Updownarrow$$

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \text{ converges.}$$

Prop 3.17b Let $\{s_n\}$ be a sequence of real numbers.

For $\limsup\limits_{n\to\infty} s_n = s^* < x$, $\exists N\ \forall n\geq N\ s_n \leq x$.

Proof Suppose $\exists$ ∞ly many $s_n > x$.

$\exists$ subsequence $\{s_{n_k}\}$ of s_n s.t. $\forall k$ $s_{n_k} > x$

$\limsup\limits_{k\to\infty} s_{n_k} \geq x$ by Thm 3.19

$x > \limsup\limits_{n\to\infty} s_n \geq \limsup\limits_{k\to\infty} s_{n_k} \geq x$ Contradiction.

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Thm: (Root test) Let $\sum_{n=1}^{\infty} a_n$ be given

$0 \le \alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then

(a) if $\alpha < 1$ then $\sum a_n$ converges
(b) if $\alpha > 1$ then $\sum a_n$ diverges
(c) if $\alpha = 1$ then the test is inconclusive.

Proof (a) Since $\alpha < 1$ $\exists \beta$ s.t. $\alpha < \beta < 1$

$$\limsup \sqrt[n]{|a_n|} = \alpha < \beta,$$

by previous prop. $\exists N$ $\forall n \ge N$ $\sqrt[n]{|a_n|} \le \beta$.

$|a_n| \le \beta^n$.

$$\underbrace{\sum_{n=N}^{\infty} a_n}_{\text{convergent}} \text{ compares to } \underbrace{\sum_{n=N}^{\infty} \beta^n}_{\text{convergent } (\beta<1)}$$

convergent $\Longleftarrow$ convergent $(\beta < 1)$

by comparison test Thm 3.25

$$\underbrace{\sum_{n=1}^{\infty} a_n = \underbrace{a_1 + \cdots + a_{N-1}}_{\text{finite sum}} + \underbrace{\sum_{n=N}^{\infty} a_n}_{\text{convergent}}}_{\text{all sum is convergent}}$$

(b) $\exists$ subsequence $\sqrt[n_k]{|a_{n_k}|} \longrightarrow \alpha > 1$.

For large k: $\sqrt[n_k]{|a_{n_k}|} \ge 1 \Rightarrow |a_{n_k}| \ge 1$

Prop.3.23 $\sum_{n=1}^{\infty} a_n$ were convergent then $a_n \to 0$, which can't happen if $|a_{n_k}| \ge 1$.

(c) $$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

$$\sqrt[n]{\frac{1}{n^p}} = \left(\frac{1}{\sqrt[n]{n}}\right)^p \longrightarrow 1, \quad (\text{regardless of } p)$$

since $\sqrt[n]{n} \to 1$, p is fixed.

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{divergent}} \qquad \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\text{convergent}}$$

$\alpha = 1$ in both.

(Ex) $$\frac{1}{1-r}$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = 1 + \frac{1}{4} + \frac{1}{16} + \cdots = \frac{4}{3}$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{9}\right)^n = 1 + \frac{1}{9} + \frac{1}{81} + \cdots = \frac{9}{8}$$

Let $$a_n = \begin{cases} (1/4)^{n/2} & \text{if } n \text{ is even} \\ (1/9)^{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

$$\sum_{n=1}^{\infty} a_n : \quad 1 + 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{81} + \frac{1}{64} + \frac{1}{729} + \cdots$$

Root test:

n even $$\sqrt[n]{|a_n|} = \left(\left(\frac{1}{4}\right)^{n/2}\right)^{\frac{1}{n}} = \frac{1}{2}$$

n odd $$\sqrt[n]{|a_n|} = \left(\left(\frac{1}{9}\right)^{\frac{n-1}{2}}\right)^{\frac{1}{n}} = \left(\frac{1}{3}\right)^{\frac{n-1}{n}} = \frac{1}{3} \cdot \underset{\substack{\downarrow \\ 1}}{\left(\frac{1}{3}\right)^{-\frac{1}{n}}}$$

$$\sqrt[n]{3} \to 1$$

$$\limsup \sqrt[n]{|a_n|} = \frac{1}{2} = \alpha < 1$$

$$\liminf \sqrt[n]{|a_n|} = \frac{1}{3}$$

$\Longrightarrow$ $\sum a_n$ converges. (By root test).

Exam MT1.

Median 78

Average 69

A	84	53%
A-	78	
B+	72	11%
B	66	
B-	60	
C+	54	26%
C	48	
C-	42	
D+	36	11%
D	30	
D-	24	

Post on ICON Average & Letter equivalent

MT1	22.5%
HW	4%
Qz	4%
	30.5%