March 11, 2020

①

SERIES (in $\mathbb{R}$, unless stated otherwise)

Corollary of CCPS:

Prop $\sum_{n=1}^{\infty} a_n$ is convergent $\Longrightarrow$ $\lim_{n\to\infty} a_n = 0$

Proof: CCPS: $\forall \varepsilon>0\ \exists N\ \forall m=n\geq N\ \left|\sum_{k=n}^{m} a_k\right| = |a_n| < \varepsilon$

Converse is false: $\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$, $\frac{1}{n} \to 0$.

Thm: (Comparison Test)

(a) If $|a_n| \leq c_n$, $\forall n\geq N_0$, then

(i) $\sum_n c_n$ converges $\Longrightarrow$ $\sum_n a_n$ converges.

(ii) $\sum_n c_n$ converges $\Longrightarrow$ $\sum_n |a_n|$ converges.

(b) If $a_n \geq d_n \geq 0$, $\forall n\geq N_0$, then

$$\left(\sum_n d_n \text{ diverges} \Longrightarrow \sum_n a_n \text{ diverges}\right)$$

Proof of (a) Given $\sum_n c_n$ converges.

So, $\{c_n\}$ satisfies CCPS

$\Longrightarrow \forall \varepsilon>0\ \exists N\ \forall n,m\geq N,\ m\geq n\quad \left|\sum_{k=n}^{m} c_k\right| < \varepsilon.$

$$\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} c_k = \left|\sum_{k=n}^{m} c_k\right| < \varepsilon.$$

since $c_k \geq 0$

Conclusion: $\{|a_k|\}$ and $\{a_k\}$ both satisfy CCPS.

Both $\{|a_k|\}$ and $\{a_k\}$ are convergent series.

(b) $a_n \geq d_n \geq 0$

$$\left|\sum_{k=n}^{m} a_k\right| = \sum_{k=n}^{m} a_k \geq \sum_{k=n}^{m} d_k = \left|\sum_{k=n}^{m} d_k\right|$$

$\sim$ CCPS for $\{d_n\}$

$$\sim\left(\forall \varepsilon>0\ \exists N\ \forall\ m\geq n\geq N \quad \left|\sum_{k=n}^{m} d_k\right| < \varepsilon\right)$$

$$\exists \varepsilon>0\ \forall N\ \exists\ m, n, > N \quad \left|\sum_{k=n}^{m} d_k\right| \geq \varepsilon$$

$$\Longrightarrow \quad \exists \varepsilon>0\ \forall N\ \exists\ m\geq n\geq N \quad \left|\sum_{k=n}^{m} a_k\right| \geq \varepsilon.$$

Corollary $\sum_{n=1}^{\infty} |a_n|$ converges $\Longrightarrow \sum_{n=1}^{\infty} a_n$ converges.

In (a) take $|a_n| \leq c_n = |a_n|$

Series with non-negative terms $\quad a_n \geq 0$

$$S_{n+1} = S_n + a_{n+1} \geq S_n \geq 0$$

Recall $\underbrace{\{S_n\} \text{ converges } \Longleftrightarrow S_n \text{ is bounded}}_{\text{for all monotone sequences.}}$

Examples

Defn $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = e$ $\left(\text{prop } e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \right)$

Why is this $\uparrow$ convergent?

① $\frac{1}{n!} \geq 0$

② $S_n = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots + \frac{1}{n!}$

$\leq 1 + 1 + \underbrace{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}}}_{\leq 1} \leq 3$

$\forall n$ $0 \leq S_n \leq 3$, $S_n \uparrow$. $\Rightarrow$ S_n converges.
define $\lim S_n := e$.

Ex if $-1 < x < 1$, then $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

Proof $S_n = 1 + x + \cdots + x^n = \frac{1-x^{n+1}}{1-x} \longrightarrow \frac{1}{1-x}$.

since $|x| < 1 \Rightarrow |x|^n \to 0$ as $n \to \infty$

Not from Calc II:

Thm 3.27

Let $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_k \geq a_{k+1} \geq 0 \quad \forall k.$

$$\sum_{n=0}^{\infty} a_n \text{ converges} \iff \sum_{k=0}^{\infty} 2^k a_{2^k} \text{ converges.}$$

Applications ① $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\iff p > 1$

Proof:

Take $a_n = \dfrac{1}{n^p}$

$$2^k a_{2^k} = 2^k \frac{1}{(2^k)^p} = \frac{2^k}{2^{kp}} = \frac{1}{2^{k(p-1)}} = \frac{1}{(2^{p-1})^k}$$

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = \sum_{k=0} \left(\frac{1}{2^{p-1}}\right)^k = \sum_{k=0}^{\infty} r^k \quad \text{where}$$

$$r = \frac{1}{2^{p-1}}.$$

Recall $r > 0 \left(\displaystyle\sum_{k=0}^{\infty} r^k \text{ converges} \iff r < 1 \right)$

$$\frac{1}{2^{p-1}} < 1 \iff 2^{p-1} > 1 \iff p > 1.$$

Hence

$$\sum_{n=0}^{\infty} \frac{1}{n^p} \text{ converges} \iff \sum_{k=0}^{\infty} \frac{1}{(2^{p-1})^k} \text{ converges} \iff p > 1.$$

Proof of 3.27 $\quad a_i \geq 0 \ \forall i$

$0 \leq s_n = a_1 + a_2 + \cdots + a_n \uparrow$

$0 \leq t_k = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots + 2^k a_{2^k} \uparrow.$

(i) WTS: for $n < 2^k$, $s_n \leq t_k$. ⊛

$$s_n \leq a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) +$$
$$+ (a_8 + a_9 + \cdots + a_{15}) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1})$$

$$\leq a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots + 2^k a_{2^k} = t_k.$$

since: $a_3 \leq a_2$ $\quad a_4 \geq a_5 \geq a_6 \geq a_7$ $\quad$ This proves ⊛

If $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges $\Rightarrow$ t_k is bounded, $\uparrow$.

$\forall k \ t_k \leq M$. for some M.

$\Rightarrow$ s_n is bounded & $\uparrow$

$s_n \leq t_k \leq M$

$\Rightarrow$ s_n converges

$\Rightarrow \sum_{n=1}^{\infty} a_n$ converges.

Claim:

(ii) For $n > 2^k$, $2s_n \geq t_k$

$$s_n \geq a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k})$$

$a_3 \leq a_4$

$$\geq a_1 + a_2 + 2a_4 + 4a_8 + \cdots + 2^{k-1}a_{2^k}$$

$$\geq \frac{1}{2}a_1 + a_2 + 2a_4 + \cdots + 2^{k-1}a_{2^k} = \frac{1}{2}t_k.$$

$$s_n \geq \frac{1}{2}t_k \quad \text{for} \quad n > 2^k$$

If $\sum a_n$ converges then s_n is a bounded, $\uparrow$ sequence

$\implies t_k$ is " " $\uparrow$ "

$\implies \sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.