

①

Lim sup / lim inf

Given  $\{s_n\}$  a sequence in  $\mathbb{R}$ 

$$E = \{x \in \mathbb{R} \cup \{\infty, -\infty\} \mid \exists \{s_{n_k}\} \text{ a subsequence of } \{s_n\} \text{ s.t. } s_{n_k} \rightarrow x\}$$

- $E$  is closed.
- If  $\sup E < \infty$ , and if  $\inf E > -\infty$  then  $s_n$  is a bounded sequence,  $\exists s_{n_k} \rightarrow s^*$   
 $\exists s_{n'_k} \rightarrow s_*$

where  $\sup E = s^*$  $\inf E = s_*$ 

- If  $\{s_n\}$  is unbounded then  $\exists$  there exists a subsequence  $\longrightarrow s^*$ , and also a subsequence  $\longrightarrow s_*$ . ( $\pm\infty$  possible)

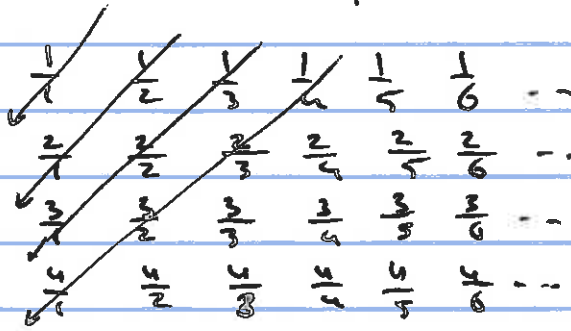
lim inf & lim sup are attainable as limits of some subsequences.

$$\Rightarrow s_n = \left( (-1)^n + 1 \right) \frac{n}{n+1}$$

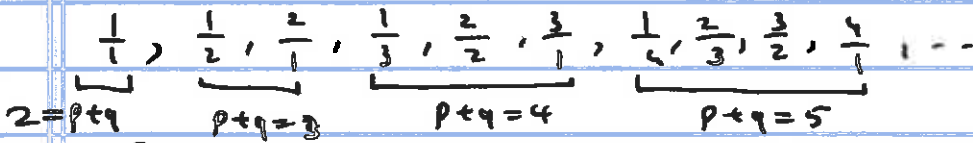
$$\liminf s_n = 0$$

$$\limsup s_n = 2$$

list all positive rational numbers in an array



List the diagonals  $p+q=n$  as  $n=2, 3, 4, \dots$



Diagonally take all  $\frac{p}{q}$  without discarding any. every positive rational number  $\frac{p}{q}$  appears in this sequence infinitely many times

$$\frac{p}{q}, \frac{2p}{2q}, \frac{3p}{3q}, \dots, \frac{np}{nq}, \dots$$

Claim  $E = [0, \infty] \subseteq \mathbb{R}$  is the set of all subsequential limits of  $S_n$

$$\liminf S_n = 0 \leftarrow \left\{ \frac{1}{n} \right\}$$

$$\limsup S_n = +\infty \leftarrow \{n\}$$

Let  $x \in (0, \infty)$  be any real number.

$$\text{then } \exists p_n \in \mathbb{Q}^+ \quad x - \frac{1}{n} < p_n < x + \frac{1}{n}$$

$$\{p_n\} \rightarrow x$$

(WTS)  $\{p_k\}$  is a subsequence  $\{s_{n_k}\}$  of  $\{s_n\}$ .

Choose  $p_1 = s_{n_1}$

Assume  $p_1 \dots p_k$  are chosen

$$p_i = s_{n_i} \quad 1 \leq i \leq k.$$

Since  $p_{k+1}$  appears in  $\{s_n\}$  infinitely many times  $\exists n_{k+1}$  s.t.  $p_{k+1} = s_{n_{k+1}}$ .

s.t.  $n_{k+1} > n_k$

$$s_{n_k} \rightarrow x. \quad \forall x \in [0, \infty] \exists s_{n_k} \rightarrow x.$$

$\therefore E = [0, \infty].$

Q.  $\exists$ ? sequence for which  $E = \mathbb{R}$ ?

$$\frac{1}{1}, -\frac{1}{1}, \frac{1}{2}, -\frac{1}{2}, \frac{2}{1}, -\frac{2}{1}, \frac{1}{3}, -\frac{1}{3}, \dots$$

Take previous sequence, take each rational number  $\frac{p}{q}$ , followed by  $-\frac{p}{q}$ .

Thm 3.19 If  $s_n \leq t_n \quad \forall n \geq N$  for some  $N$  (fixed) then

- (i)  $\liminf s_n \leq \liminf t_n$
- (ii)  $\limsup s_n \leq \limsup t_n$

Given  $s_n \leq t_n$  want  $\limsup s_n \leq \limsup t_n$

Proof of (ii)

Case 1 •  $\limsup s_n = -\infty$

•  $\limsup t_n \text{ exists } \geq -\infty$

Case 2 •  $\limsup s_n = +\infty$

•  $\exists$  subsequence  $s_{n_k} \rightarrow +\infty$ .

$\forall M \exists k. \forall n \geq k \quad s_n \geq M$

$t_{n_k} \geq s_{n_k} \geq M$

$t_{n_k} \rightarrow +\infty$

$\limsup s_n = +\infty = \limsup t_n$

Case 3 •  $\limsup s_n = s^* \in \mathbb{R}$

•  $\exists s_{n_k} \rightarrow s^*$  as  $k \rightarrow \infty$ .

$t_{n_k} \geq s_{n_k} \rightarrow s^*$

$t_{n_k}$  may not converge.

but,  $\limsup_{k \rightarrow \infty} t_{n_k} = t^*$  exists.

$\exists$  subsequence  $t_{n_{k_l}} \rightarrow t^*$  as  $l \rightarrow \infty$

$t_{n_{k_l}} \geq s_{n_{k_l}} \rightarrow s^*$  (since  $s_{n_k} \rightarrow s^*$ )

$t^*$

$t^* \geq s^*$



# SERIES

Def

Given  $\{a_n\}_{n=1}^{\infty}$ , in  $\mathbb{R}$  or  $\mathbb{C}$

$$S_n = \sum_{k=1}^n a_k = a_1 + a_2 + \dots + a_n \quad \text{is called}$$

the partial sums for  $\{a_n\}$ .

Def We say that  $\sum_{n=1}^{\infty} a_n$  converges if

$$S_n = \sum_{k=1}^n a_k \rightarrow S \quad \text{as } n \rightarrow \infty.$$

(in  $\mathbb{R}$ ) Prop (CCPS) Cauchy criterion for Partial sums.

$$\sum_{n=1}^{\infty} a_n \text{ is convergent} \iff \left\{ \begin{array}{l} \forall \epsilon > 0 \exists N \in \mathbb{N} \\ \forall n, m \geq N, m \geq n \\ \left| \sum_{k=n}^m a_k \right| < \epsilon \end{array} \right.$$

Proof

$\sum_{n=1}^{\infty} a_n$  is convergent

$$\iff S_n = \sum_{k=1}^n a_k \text{ is convergent}$$

$\iff \{S_n\}$  is Cauchy ( $\mathbb{R}$  complete)

$$\iff \forall \epsilon > 0 \exists N \quad \forall n, m \geq N, m \geq n \quad |S_m - S_{n-1}| < \epsilon$$

$$\iff \quad \quad \quad \quad \quad \quad \quad \quad \left| \sum_{k=1}^m a_k - \sum_{k=1}^{n-1} a_k \right| < \epsilon$$

$$\iff \quad \quad \quad \quad \quad \quad \quad \quad \left| \sum_{k=n}^m a_k \right| < \epsilon.$$