March 9, 2020

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Lim sup / liminf

Given $\{s_n\}$ a sequence in $\mathbb{R}$

$E = \{ x \in \mathbb{R} \cup \{\infty, -\infty\} \mid \exists \{s_{n_k}\}$ a subsequence of $\{s_n\}$ s.t. $s_{n_k} \to x \}$

- E is closed.
- If $\sup E < \infty$, and if $\inf E > -\infty$ then s_n is a bounded sequence, $\exists\, s_{n_k} \to s^*$, $\exists\, s_{n_k'} \to s_*$

  where $\sup E = s^*$, $\inf E = s_*$
- If $\{s_n\}$ is unbounded then $\exists$ there exists a subsequence $\longrightarrow s^*$, and also a subsequence $\longrightarrow s_*$. ($\pm\infty$ possible)

liminf & limsup are attainable as limits of some subsequences.

Ex $\quad s_n = \left((-1)^n + 1\right) \dfrac{n}{n+1}$

$\liminf s_n = 0$

$\limsup s_n = 2$

Ex: list all positive rational numbers in an array

$$\begin{array}{cccccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \frac{2}{6} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \frac{3}{6} & \cdots \\
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \frac{4}{6} & \cdots \\
\vdots
\end{array}$$

List the diagonals $p+q=n$ as $n=2,3,4,\dots$

$$\underbrace{\frac{1}{1}}_{2=p+q},\ \underbrace{\frac{1}{2},\frac{2}{1}}_{p+q=3},\ \underbrace{\frac{1}{3},\frac{2}{2},\frac{3}{1}}_{p+q=4},\ \underbrace{\frac{1}{4},\frac{2}{3},\frac{3}{2},\frac{4}{1}}_{p+q=5},\ \dots$$

Diagonally take all $\frac{p}{q}$ <u>without discarding</u> any. every positive rational number $\frac{p}{q}$ appears in this sequence infinitely many times

$$\frac{p}{q} \quad \frac{2p}{2q} \quad \frac{3p}{3q} \ \cdots \ \frac{np}{nq} \ \cdots$$

<u>Claim</u> $E = [0,\infty]$ $\subseteq\mathbb{R}$ is the set of all subsequential limits of s_n

$$\liminf s_n = 0 \longleftarrow \{\tfrac{1}{n}\}$$
$$\limsup s_n = +\infty \longleftarrow \{n\}$$

Let $x \in (0,\infty)$ be any real number.

$\forall n \in \mathbb{N}$ $\exists\, p_n \in \mathbb{Q}^+$ $\quad x - \frac{1}{n} < p_n < x + \frac{1}{n}$

$$\{p_n\} \longrightarrow x$$

(WTS) $\{p_k\}$ is a subsequence $\{s_{n_k}\}$ of $\{s_n\}$.

Choose $p_1 = s_{n_1}$

Assume $p_1, \cdots p_k$ are chosen

$$p_i = s_{n_i} \quad 1 \le i \le k.$$

Since p_{k+1} appears in $\{s_n\}$ infinitely many times $\exists\, n_{k+1}$ s.t. $p_{k+1} = s_{n_{k+1}}$

s.t. $n_{k+1} > n_k$.

$s_{n_k} \to x$. $\forall x \in [0, \infty]\ \exists\, s_{n_k} \to x$.

So $E = [0, \infty]$.

Q. $\exists$? sequence for which $E = \mathbb{R}$?

$\frac{1}{1}, -\frac{1}{1}, \frac{1}{2}, -\frac{1}{2}, \frac{2}{1}, \frac{-2}{1}, \frac{1}{3}, \frac{-1}{3} \cdots$

Take previous sequence, take each rational number $\frac{p}{q}$, followed by $-\frac{p}{q}$.

<u>Thm 3.19</u> If $s_n \le t_n$ $\forall n \ge N$ for some N (fixed) then

(i) $\liminf s_n \le \liminf t_n$

(ii) $\limsup s_n \le \limsup t_n$

Given $s_n \le t_n$ want $\limsup s_n \le \limsup t_n$

Proof of (ii)

Case 1 • $\limsup s_n = -\infty$

$\limsup t_n$ exists. $\ge -\infty$.

Case 2 • $\limsup s_n = +\infty$

$\exists$ subsequence $s_{n_k} \to +\infty$.

$\forall M \; \exists K. \; \forall k \ge K \quad s_{n_k} \ge M$

$t_{n_k} \ge s_{n_k} \ge M$

$t_{n_k} \to +\infty$.

$\limsup s_n = +\infty = \limsup t_n$.

Case 3 • $\limsup s_n = \overset{\max E}{\overset{\shortparallel}{s^*}} \in \mathbb{R}$.

$\exists \quad s_{n_k} \longrightarrow s^* \quad$ as $k \to \infty$.

$t_{n_k} \ge s_{n_k} \longrightarrow s^*$

t_{n_k} may not converge.

but, $\limsup\limits_{k\to\infty} t_{n_k} = t^*$ exists.

$\exists$ subsequence $t_{n_{k_\ell}} \longrightarrow t^* \quad$ as $\ell \to \infty$

$$\begin{array}{l} t_{n_{k_\ell}} \ge s_{n_{k_\ell}} \longrightarrow s^* \quad (\text{since } s_{n_k} \to s^*) \\ \downarrow \\ t^* \end{array}$$

$t^* \ge s^*$.

why?

$$s^* \le t^* \le t^{**}$$

$\limsup\limits_{n} s_n \qquad \limsup\limits_{k} t_{n_k} \qquad \limsup\limits_{n} t_n$

t_{n_k} is a subsequence of t_n.

Each subsequence of a subsequence is a subsequence of the original.

E = limits of all convergent subsequences of t_n

E_0 = " " " " " " t_{n_k}

$$E_0 \subseteq E$$

$$t^* = \max/\sup E_0 \le \max/\sup E = t^{**}$$

$$\limsup s_n = s^* \le t^{**} = \limsup t_n$$

<u>Prop</u>

$$\lim_{n\to\infty} \frac{1}{n^p} = 0 \qquad \text{if } p > 0$$

$$\lim_{n\to\infty} \sqrt[n]{p} = 1 \qquad \text{if } p > 0$$

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

$$\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0 \qquad \alpha \in \mathbb{R},\ p > 0$$

$$\lim_{n\to\infty} |x|^n = \begin{cases} 0 & \text{if } |x| < 1 \\ 1 & \text{if } |x| = 1 \\ \infty & \text{if } |x| > 1. \end{cases}$$

$$\lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } -1 < x < 1 \\ 1 & \text{if } x = 1 \\ \text{divergent} & \text{if } x \le -1 \\ \infty & \text{if } x > 1 \end{cases}$$

SERIES

Defn Given $\{a_n\}_{n=1}^{\infty}$, in $\mathbb{R}$ or $\mathbb{C}$

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \dots + a_n$$ is called

the partial sums for $\{a_n\}$.

Def We say that $\sum_{n=1}^{\infty} a_n$ converges if

$$s_n = \sum_{k=1}^{n} a_k \longrightarrow s \quad \text{as} \quad n \to \infty .$$

(in $\mathbb{R}$) **Prop** (CCPS) Cauchy criterion for Partial sums.

$$\sum_{n=1}^{\infty} a_n \text{ is convergent} \iff \begin{cases} \forall \varepsilon > 0 \ \exists N \in \mathbb{N} \\ \forall n, m \geq N, \ m \geq n \\ \left| \sum_{k=n}^{m} a_k \right| < \varepsilon \end{cases}$$

Proof

$\sum_{n=1}^{\infty} a_n$ is convergent

$\iff$ $s_n = \sum_{k=1}^{n} a_k$ is convergent

$\iff$ $\{s_n\}$ is Cauchy ($\mathbb{R}$ complete)

$\iff$ $\forall \varepsilon > 0 \ \exists N \quad \forall n, m \geq N, \ m \geq n \quad |s_m - s_{n-1}| < \varepsilon$

$\iff$ " " " " $\left| \sum_{k=1}^{m} a_k - \sum_{k=1}^{n-1} a_k \right| < \varepsilon$

$\iff$ " " " " $\left| \sum_{k=n}^{m} a_k \right| < \varepsilon .$