March 4, 2020

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Review Thursday 5:00 – 6:30

MONOTONE SEQUENCES

Def. A sequence $\{s_n\}$ in $\mathbb{R}$ is called monotone if

- $\forall n \quad s_{n+1} \geq s_n$, (monotone increasing)

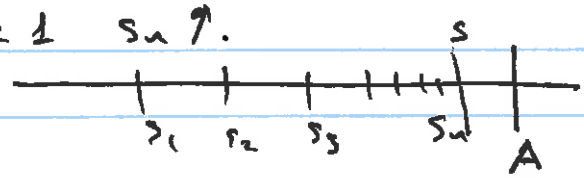
OR
- $\forall n \quad s_{n+1} \leq s_n$ (monotone decreasing)

Thm Let $\{s_n\}$ be a monotone sequence.

$\{s_n\}$ is convergent $\iff$ $\{s_n\}$ is bounded.

Proof: ($\Rightarrow$) True for every sequence.

($\Leftarrow$:) Case 1 $s_n \uparrow$.



$S = \{s_n \mid n \in \mathbb{N}\}$ is bounded. s_1 lower bound

$\exists A \in \mathbb{R}, \ \forall n \in \mathbb{N} \ s_n \leq A$

LUB $\Rightarrow \exists \ s = \sup \{s_n \mid n \in \mathbb{N}\}$.

Let $\varepsilon > 0$ be given $s - \varepsilon$ is not an upper bd for S.

$\exists \ s_N$ s.t. $\quad s - \varepsilon < s_N \leq s$

$\forall n \geq N \quad s - \varepsilon < s_N \leq s_n \leq s < s + \varepsilon$

$|s_n - s| < \varepsilon$ # since $\lim s_n = s$.

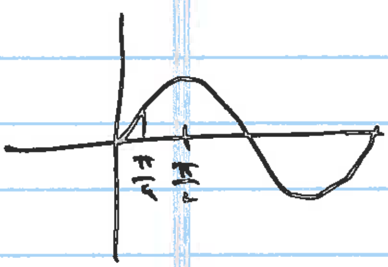
Thm: Let $\{p_n\}_{n=1}^{\infty}$ be a sequence in any metric space (X, d).

$$L = \{ p \in X \mid \exists \text{ a subsequence } \{p_{n_k}\}_{k=1}^{\infty} \text{ of } \{p_n\}_{n=1}^{\infty} \text{ s.t. } p_{n_k} \to p \}$$

Then L is closed.

Ex $\{t_n\} = \sin \frac{n\pi}{4}$

$\frac{\sqrt{2}}{2}$ 1 $\frac{\sqrt{2}}{2}$ 0 $-\frac{\sqrt{2}}{2}$, -1, $-\frac{\sqrt{2}}{2}$, 0, ...



$$L = \{ \frac{\sqrt{2}}{2}, 1, -\frac{\sqrt{2}}{2}, 0, -1 \}$$

Main idea of the proof $L' \subseteq L$ want

$p_{n_1}, p_{n_2}, \ldots \longrightarrow q_1 \in L$

$p_{n_1'}, p_{n_2'}, \ldots \longrightarrow q_2 \in L$

$\vdots$

$q_n \in L$

$\downarrow$

$q \in L'$

<u>Proof.</u> Want to show $\forall q \in L'$, $q \in L$; i.e. $L' \subseteq L$. to obtain L is closed.

Fix $q \in L'$.

$$\forall k \in \mathbb{N} \quad \exists q_k \in N_{\frac{1}{2k}}(q) \cap (L - \{q\}) \neq \phi$$

For each q_k, there exists a subsequence of $\{p_n\}$, s.t. (subseq.) converging to q_k, since $q_k \in L$

$q_1 \in L$ $\exists$ subsequence of $\{p_n\}$ $\longrightarrow q_1$

$N_{\frac{1}{2}}(q_1)$ contains p_n for infinitely many n.

$\exists p_{m_1} \in N_{\frac{1}{2}}(q_1) \subseteq N_1(q)$, since $q_1 \in N_{\frac{1}{2}}(q)$

$q_2 \in L$, $\exists$ subsequence of $\{p_n\}$ $\longrightarrow q_2$.

$\exists p_{m_2} \in N_{\frac{1}{4}}(q_2) \subseteq N_{\frac{1}{2}}(q)$, since $q_2 \in N_{\frac{1}{4}}(q)$

I can choose $m_2 > m_1$, since $N_{\frac{1}{4}}(q_2)$ contains p_n for infinitely many n

Inductively:

$\exists p_{m_k} \in N_{\frac{1}{2k}}(q_k) \subseteq N_{\frac{1}{k}}(q)$ since $q_k \in N_{\frac{1}{2k}}(q)$

since it contains p_n for infinitely many n, so we can take

$$m_1 < m_2 < m_3 < \cdots < m_k$$

$\forall k$ $p_{m_k} \in N_{\frac{1}{k}}(q)$, $d(p_{m_k}, q) < \frac{1}{k}$ $\Longrightarrow$ $\lim_{k\to\infty} p_{m_k} = q \in L$.

<u>Defn</u> Let $\{s_n\}$ be a sequence in $\mathbb{R}$.

If $\forall M \in \mathbb{R}\ \exists N \in \mathbb{N}\ \forall n \geq N,\quad s_n \geq M$, then

we say $s_n \to +\infty$,

or s_n diverges to $+\infty$.

If $\forall M \in \mathbb{R}\ \exists N \in \mathbb{N}\ \forall n \geq N\quad s_n \leq M$, then

we say $s_n \to -\infty$

or s_n diverges to $-\infty$.

<u>Defn</u> Let $\{s_n\}$ be a sequence in $\mathbb{R}$.

Let $E = \{ x \in \mathbb{R} \cup \{\infty, -\infty\} \mid \exists$ a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ s.t. $s_{n_k} \to x \}$

Define $\limsup\limits_{n\to\infty} s_n = \overline{\lim\limits_{n\to\infty}}\, s_n = \sup E = s^*$

$\liminf\limits_{n\to\infty} s_n = \underline{\lim\limits_{n\to\infty}}\, s_n = \inf E = s_*$.

Why? PTO $\rightarrow$ Since $E \neq \emptyset$, $\sup E$ and $\inf E$ always exist, in $\mathbb{R} \cup \{\infty, -\infty\}$

$\limsup s_n \in \mathbb{R} \cup \{-\infty, \infty\}$

$\liminf s_n \in \mathbb{R} \cup \{-\infty, \infty\}$

Pages 5 + $5\frac{1}{2}$ are rewritten

Why $E \neq \phi$?

Recall Bolzano-Weierstrass:
If a sequence $\{s_n\}$ in $\mathbb{R}$ is bounded, then it has a convergent subsequence (with a limit in $\mathbb{R}$). $E \neq \phi$.

If a sequence $\{s_n\}$ in $\mathbb{R}$ is not bounded
i.e. not $(\exists M \, \forall n \in \mathbb{N}, \ |s_n| \leq M)$
$\forall M \, \exists n$ s.t. $|s_n| > M$.

Take $M = k \in \mathbb{N}$, then $\exists \, n(k)$ s.t. $|s_{n(k)}| > k$.
i.e. $s_{n(k)} > k$ <u>OR</u> $s_{n(k)} < -k$.

Let
$\mathrm{I} = \{ n(k) \mid s_{n(k)} > k \}$ $\mathrm{I} \cup \mathrm{II} \subseteq \mathbb{N}$.
$\mathrm{II} = \{ n(k) \mid s_{n(k)} < -k \}$.

If both I and II were finite, there is a maximal k_1, with no $s_n > k_1$, and a maximal k_2 with no $s_n < -k_2$.
that is $-k_2 \leq s_n \leq k_1$, i.e. bounded.
But $\{s_n\}$ is unbounded.
So either I is infinite or II is infinite. (Both is possible)

<u>CRUCIAL</u>: $s_{n(k)}$ is not necessarily a subsequence of s_n.

PTO

p $5\frac{1}{2}$

But we can extract a subsequence $\{s_{n_\ell}\}$ of $\{s_n\}$ s.t. either $s_{n_\ell} \longrightarrow +\infty$ <u>OR</u> $s_{n_\ell} \longrightarrow -\infty$.

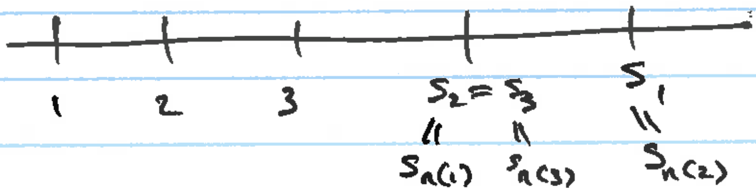
Then either $+\infty$ or $-\infty \in E$, and $E \neq \phi$.

We will do it for I infinite. (II infinite case will be similar.)

(*) • $S_{n(k)} > k$, • $n(k)$ may be repeated for different k

• $S_{n(k+1)} > k+1$
$S_{n(k)} > k$ $\not\Rightarrow$ $n(k+1) > n(k)$

$S_{n(k)}$ is not a subsequence of S_n



How does one get a subsequence from I, if I has infinitely many $n(k)$?

Answer: Choose S_{m_1} to be any $S_n > 1$.

Assume (Inductively) $S_{m_1}, S_{m_2}, \dots S_{m_\ell}$ chosen s.t.

• $m_1 < m_2 < m_3 < \dots < m_\ell$ and

• $S_{m_i} > i$ for $i = 1, 2, \dots \ell$.

We choose $S_{m_{\ell+1}}$ as follows:

Take $a \in \mathbb{N}$ s.t.

$$a \geq \max\left(\ell, \max_{1 \leq j \leq m_\ell} |S_j|\right) + 1.$$

Find $n(a)$ from I i.e. $S_{n(a)} > a$

$$S_{n(a)} > a > |S_j| \text{ for all } j = 1, 2, \dots, m_\ell$$

$n(a) > m_\ell$

Choose $n(a) = m_{\ell+1}$, hence $m_{\ell+1} > m_\ell$.

$S_{m_{\ell+1}} > a \geq \ell + 1$.

• $\{S_{m_\ell}\}_{\ell=1}^{\infty}$ is a subsequence of $\{S_n\}$

• $\lim_{\ell \to \infty} S_{m_\ell} = +\infty$.

- $s_n = ((-1)^n + 1)n$

  $s_n \quad 0, 2, 0, 4, 0, 6, 0, 8, \ldots$

  $E = \{0, +\infty\}$ $\qquad$ $\liminf s_n = 0$

  $\limsup s_n = \infty.$

- $t_n = (-1)^n \cdot n$

  $E = \{-\infty, \infty\}$ $\qquad$ $\limsup t_n = \infty$

  $\liminf t_n = -\infty.$