March 2, 2020

CAUCHY SEQUENCES & COMPLETENESS

Def^n A sequence $\{p_n\}_{n=1}^{\infty}$ in a metric space (X, d) is called Cauchy if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}, \ \forall n, m \geq N \quad d(p_n, p_m) < \varepsilon.$$

Lemma: Every convergent sequence is Cauchy. (in all metric spaces)

Proof Let $p_n \to p$ for $\{p_n\}_{n=1}^{\infty}$ in (X, d)

$$\forall \varepsilon > 0 \ \exists N \ \forall n \geq N \quad d(p_n, p) < \frac{\varepsilon}{2}.$$

$$\forall n, m \geq N \quad d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

#.

Example $\sqrt{2} \in \mathbb{R}$, $\forall n \in \mathbb{N} \ \exists \ p_n \in \mathbb{Q} \cap (\sqrt{2} - \frac{1}{n}, \sqrt{2} + \frac{1}{n})$
Density of $\mathbb{Q}$ in $\mathbb{R}$.

$p_n \to \sqrt{2}$ since

$$\forall \varepsilon > 0 \ \exists N \quad N > \frac{1}{\varepsilon}, \ \forall n \geq N$$

$$|\sqrt{2} - p_n| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

(Lemma: $\Rightarrow$) $\{p_n\}$ is Cauchy in $\mathbb{R}$.

Look at the
Same sequence in $\mathbb{Q}$.

$$|p_n - p_m|_{\mathbb{Q}} = |p_n - p_m|_{\mathbb{R}}.$$

$\{p_n\}$ is Cauchy in $\mathbb{Q}$. ($p_n \in \mathbb{Q}$, but $p \notin \mathbb{Q}$)

$\{p_n\}$ doesn't converge in $\mathbb{Q}$.

Cauchy $\not\Rightarrow$ Convergent in $\mathbb{Q}$.

Defn A metric space (X, d) is called complete if every Cauchy sequence in (X,d) is convergent.

***** Thm (1) Every compact metric space is complete

(2) ($\mathbb{R}^k$, standard metric) is complete, $\forall k \in \mathbb{N}$.

Prep Lemmas: Recall $\operatorname{diam}(A) = \sup\{d(x,y) \mid x, y \in A\}$, if $A \neq \emptyset$.

If (X, d) is a ~~metric~~ space.

(a) If $\bar{E}$ is the closure of E, then
$$\operatorname{diam}(E) = \operatorname{diam}(\bar{E})$$

(b) If K_n is compact $\forall n \in \mathbb{N}$, $\emptyset \neq K_{n+1} \subseteq K_n$, $\forall n$, and $\lim_{n\to\infty} \operatorname{diam}(K_n) = 0$

then $$\bigcap_{n=1}^{\infty} K_n = \{p\}.$$

Proof (a)

$$A \subseteq B \implies \operatorname{diam}(A) \leq \operatorname{diam} B$$

Why? $\{d(x,y) \mid x,y \in A\} \subseteq \{d(x,y) \mid x,y \in B\}$.

$$E \subseteq \bar{E} = E \cup E'$$

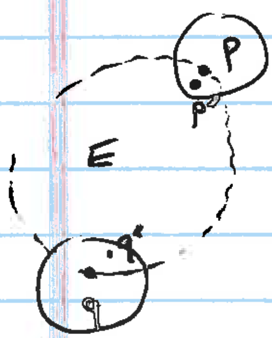
① $\operatorname{diam}(E) \leq \operatorname{diam}(\bar{E})$.

Let $p,q \in \bar{E}$ be arbitrary.

Let $\varepsilon > 0$ be given

$\exists\ p', q' \in E$ s.t. $d(p,p') < \frac{\varepsilon}{2}$

$d(q,q') < \frac{\varepsilon}{2}$



$$d(p,q) \leq d(p,p') + d(p',q') + d(q',q)$$

$$\leq \frac{\varepsilon}{2} + \operatorname{diam}(E) + \frac{\varepsilon}{2}$$

$\forall\, p,q \in \bar{E}$ $\quad d(p,q) \leq \varepsilon + \operatorname{diam}(E)$

supremum as $p,q \in \bar{E}$

$\operatorname{diam}(\bar{E}) \leq \varepsilon + \operatorname{diam}(E)$, $\forall \varepsilon > 0$

② $\operatorname{diam}(\bar{E}) \leq \operatorname{diam}(E)$

$\operatorname{diam}(E) = \operatorname{diam}(\bar{E})$ by ① + ②

Lemma (b) $\emptyset \neq K_{n+1} \subseteq K_n$ compact, $\forall n \in \mathbb{N}$

(Thm 2.36) <u>STEP 1</u> To show $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$.

Let $G_n = X - K_n$ open. $\forall n$.

$\uparrow$ compact $\to$ closed

$X \neq G_{n+1} \supseteq G_n$. Let $K = \bigcap_{n=1}^{\infty} K_n$.

Suppose K is $\emptyset$.

$$K_1 \subseteq X = X - K = X - \left(\bigcap_{n=1}^{\infty} K_n\right)$$

$$= \bigcup_{n=1}^{\infty} (X - K_n) = \bigcup_{n=1}^{\infty} G_n.$$

$\uparrow$ open.

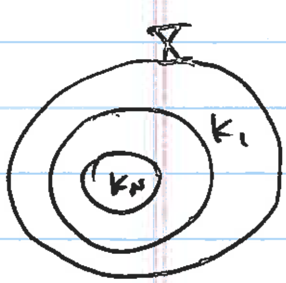
$\{G_n\}$ is an open cover of K_1, compact.

$\exists$ finite subcover $K_1 \subseteq \bigcup_{i=1}^{\ell} G_{n_i} = G_N$

$N = \max(n_1, n_2, \dots n_\ell)$

$\emptyset = K_1 - G_N = K_1 - (X - K_N) = K_N \neq \emptyset$. Contradiction

since $K_N \subseteq K_1 \subseteq X$.



This proves $K = \bigcap_{n=1}^{\infty} K_n \neq \emptyset$.

Lemma (6)

STEP 2

Suppose $\exists\, p, q \in \bigcap_{n=1}^{\infty} K_n$, $p \neq q$.

then $p, q \in K_n$. $d(p,q) \leq \operatorname{diam} K_n$.

$0 < d(p,q) \leq \operatorname{diam}(K_n) \to 0$

Hypothesis $\lim_{n\to\infty} \operatorname{diam} K_n = 0$.

Contradiction.

Hence $\bigcap_{n=1}^{\infty} K_n = \{p\}$ contains exactly one point.

***** **Proof of:** Every compact metric space is complete

(i.e. Cauchy $\Rightarrow$ convergent.)

Let $\{p_n\}_{n=1}^{\infty}$ be a Cauchy sequence in (X, d) which is a compact metric space.

<u>WANT</u> $p_n \to p$ for some $p \in X$.

$\{p_n\}_{n=1}^{\infty}$ Cauchy

Caution: N free

No particular for ε

$$\forall \varepsilon > 0 \ \exists N_0 \in \mathbb{N} \ \forall n, m \geq N_0 \ d(p_n, p_m) < \varepsilon.$$

Let $E_N = \{p_n \mid n \geq N\}$ for each $N \in \mathbb{N}$.

$$E_{N+1} \subseteq E_N \Rightarrow \overline{E_{N+1}} \subseteq \overline{E_N} \quad \forall N \in \mathbb{N}$$

$$\operatorname{diam}(\overline{E_{N_0}}) = \operatorname{diam}(E_{N_0}) \leq \varepsilon$$

Hence $\forall \varepsilon > 0 \ \exists N_0 \in \mathbb{N}$ s.t. $\operatorname{diam}(\overline{E_{N_0}}) \leq \varepsilon$

$$\lim_{N \to \infty} \operatorname{diam}(\overline{E_N}) = 0$$

$\overline{E_N}$ closed in compact X, $\overline{E_N}$ is compact.

$\emptyset \neq E_N \subseteq \overline{E_N}$

$\overline{E_{N+1}} \subseteq \overline{E_N}$: $\bigcap_{N=1}^{\infty} \overline{E_N} = \{p\}$ for some unique $p \in X$.

$p \in \bigcap_{N=1}^{\infty} \overline{E_N}$, so $p \in \overline{E_N}$ $\forall N \in \mathbb{N}$.

Let $\varepsilon > 0$ be given

Since $\lim_{N \to \infty} \operatorname{diam}(\overline{E_N}) = 0$

use N_1 since N_0 was used on page 6

$\exists N_1$ $\forall N \geq N_1$ $\operatorname{diam}(\overline{E_N}) < \varepsilon$

$\forall n \geq N$ $p_n \in E_N \subseteq \overline{E_N}$

$p \in \overline{E_N}$

$d(p_n, p) \leq \operatorname{diam}(\overline{E_N}) < \varepsilon$,

$\forall \varepsilon > 0$ $\exists N_1$ $\forall n \geq N \geq N_1$, $d(p_n, p) < \varepsilon$.

$p_n \to p$.

Thm: To prove $\mathbb{R}^k$ is complete. next

<u>Lemma</u>: Cauchy $\Longrightarrow$ bounded (HW)

<u>Proof</u> ($\mathbb{R}^k$ is complete)

Take any Cauchy sequence $\{p_n\}$ in $\mathbb{R}^k$.

<u>By Lemma</u>: $\{p_n\}$ is bounded.

$\exists R > 0$ $\{p_n\}_{n=1}^{\infty} \subseteq B_R(\vec{0}) \subseteq \underbrace{[-R,R] \times [-R,R] \times \cdots \times [-R,R]}_{\text{compact.}}$

$\{p_n\}$ converges in $[-R,R]^k$ since it is Cauchy, in a compact metric sp.

$\wedge|$

" " " $\mathbb{R}^k$.

Hence Cauchy $\Longrightarrow$ Convergent in $\mathbb{R}^k$.