Chap III

Defn A sequence $\{p_n\}$ in (X, d) is said to converge in X if

$$\exists p \in X \ \forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \in \mathbb{N}, (n \geq N \Rightarrow d(p_n, p) < \varepsilon)$$

Notation $\lim_{n \to \infty} p_n = p$ or $p_n \to p$.

- $\{p_n\}$ is said to diverge if it doesn't converge to any $p \in X$.
- $\{p_n\}$ is said to be bounded if $\exists M \in \mathbb{R} \ \exists p_0 \in X$ s.t.

$$\forall n \in \mathbb{N} \quad d(p_n, p_0) < M.$$

i.e. $p_n \in N_M(p_0)$

Thm: Let $\{p_n\}$ be a sequence in (X, d).

(a) $p_n \to p \iff \forall R > 0 \ N_R(p)$ contains all but finitely many p_n

(b) $p_n \to p, \ p_n \to q \implies p = q$.

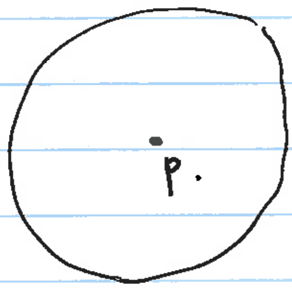
(c) $p_n \to p \implies \{p_n\}$ is bounded

(d)* $E \subseteq X \ \left(p \in E' \iff \exists \{p_n\} \text{ in } E \text{ s.t. } \begin{array}{l} p_n \neq p \ \forall n. \\ p_n \to p \end{array}\right)$

(e)** $E \subseteq X \ \left(p \in \overline{E} \iff \exists \{p_n\} \text{ in } E \text{ s.t. } p_n \to p.\right)$

Proof

(d) ($\Rightarrow$:) Assume $p \in E'$



For each $n \in \mathbb{N}$

$N_{\frac{1}{n}}(p) \cap (E - \{p\}) \neq \phi$ (Def^n E')

$\exists p_n \in N_{\frac{1}{n}}(p) \cap (E - \{p\})$.

$p_n \neq p$

$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N}$ s.t. $0 < \frac{1}{N} < \varepsilon$, $\forall n \geq N$

$d(p, p_n) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$

($\Leftarrow$:) $p_n \xrightarrow{\in E} p$, $p_n \neq p$. WTS $p \in E'$.

For $\varepsilon > 0$ $\exists N \in \mathbb{N}$ $\forall n \geq N$ $0 < d(p_n, p) < \varepsilon$

$\forall n \geq N$ $p_n \in N_\varepsilon(p) \cap (E - \{p\}) \neq \phi$. $p \in E'$.

(e) HW, recall $\bar{E} = E \cup E'$.

<u>Thm</u> Let $\{p_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^k$ (standard metric)

$$p_n = (a_{1,n}, a_{2,n}, a_{3,n} \cdots, a_{k,n}) \quad \forall n \in \mathbb{N}$$

$$p = (a_1, a_2 \cdots a_k)$$

$$\underset{\text{in } \mathbb{R}^k}{p_n \to p} \iff \lim_{n\to\infty} a_{in} \overset{\text{in } \mathbb{R}^1}{=} a_i \quad \text{for } i = 1, \cdots k.$$

<u>Proof</u>: $(\Rightarrow:)$ $\forall \varepsilon > 0 \; \exists N \; \forall n \geq N \quad |p_n - p|_{\mathbb{R}^k} < \varepsilon$

$$\varepsilon^2 > |p_n - p|^2 = \sum_{j=1}^{k} |a_{j,n} - a_j|^2 \geq |a_{j,n} - a_j|^2 \quad \text{For each } j$$

Fix j: $\forall \varepsilon > 0 \exists N \; \forall n \geq N$

$$\varepsilon > |a_{j,n} - a_j| \quad \text{i.e.} \quad a_{j,n} \to a_j. \; \left(\text{for each } j\right)$$

$(\Leftarrow:)$ $\forall j \; \forall \varepsilon > 0 \; \exists N_j \in \mathbb{N} \; \forall n \geq N_j \quad |a_{j,n} - a_j| < \frac{\varepsilon}{\sqrt{k}}$

Let $N = \max(N_1, N_2, \cdots N_k)$

$\forall n \geq N$

$$|p_n - p|^2 = \sum_{j=1}^{k} |a_{j,n} - a_j|^2 < \sum_{j=1}^{k} \left(\frac{\varepsilon}{\sqrt{k}}\right)^2 = k \frac{\varepsilon^2}{k} = \varepsilon^2$$

$$|p_n - p| < \varepsilon \quad \#.$$

Defn Given a sequence $\{p_n\}_{n=1}^{\infty}$ in X
for any increasing sequence of natural numbers
$$n_1 < n_2 < n_3 < \cdots < n_k < n_{k+1} < \cdots$$
we call $\{p_{n_k}\}_{k=1}^{\infty}$ a subsequence of $\{p_n\}_{n=1}^{\infty}$

Ex p_n : $1, 3, 5, 7, 9, \ldots$
$$p_n = 2n-1$$
$n_k = k^2$ $\quad 1 < 4 < 9 < 16 < \cdots$
$$p_{n_k} : 1, 7, 17, 31, \ldots$$
$$p_{n_k} = 2k^2 - 1$$

(3.6) **Thm:** (a) Let $\{p_n\}_{n=1}^{\infty}$ be a sequence in a compact metric space, (X, d).
Then $\exists$ a convergent subsequence $\{p_{n_k}\}_{k=1}^{\infty}$ of $\{p_n\}_{n=1}^{\infty}$.

(b) Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

Proof (a)

Let $E = \{p_n \mid n \in \mathbb{N}\}$ be the range of the sequence $\{p_n\}$.

Case 1 E is finite

Then p_n must repeat a value $p \in E$ infinitely many times

$\exists\ n_1 < n_2 < n_3 < \cdots < n_k < n_{k+1} < \cdots$ for $k \in \mathbb{N}$

s.t.

$$p_{n_k} = p$$

$$\lim_{k\to\infty} p_{n_k} = \lim_{k\to\infty} p = p.$$

Case 2 E is an infinite set, $E \subseteq (X, d)$ compact

$E' \neq \phi$ by Thm 2.37.

where $E = \{p_n \mid n \in \mathbb{N}\}$

Choose any $p \in E'$.

$$\forall n \in \mathbb{N} \qquad V_n = \underbrace{N_{\frac{1}{n}}(p) \cap (E - \{p\})}_{\text{infinitely many elements from } E \text{ Thm. 2.20}} \neq \phi$$

$k=1$ choose any $p_{n_1} \in V_1$.

$k=2$ choose $p_{n_2} \in V_2$ but $\underbrace{n_2 > n_1}$

Can do this since V_2 has infinitely many p_n.

$k=3$ " $p_{n_3} \in V_3$ s.t. $n_3 > n_2 > n_1$

same reason: V_3 has infinitely many p_n.

$\vdots$

$k=\ell$ " $p_{n_\ell} \in V_\ell$ s.t. $n_\ell > n_{\ell-1} > \cdots > n_2 > n_1$

$$d(p, p_{n_k}) < \frac{1}{k}. \quad \forall k = 1, 2, \ldots \in \mathbb{N}$$

$p_{n_k} \longrightarrow p$ since $\forall \varepsilon > 0 \ \exists N$ s.t. $\frac{1}{N} < \varepsilon$

$\forall k \geq N \quad d(p_{n_k}, p) < \frac{1}{k} < \frac{1}{N} < \varepsilon$

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<u>Proof of (b)</u> (a) + Heine Borel $\Longrightarrow$ (b)

$\{p_n\}$ bounded $\Longrightarrow \exists R > 0 \quad \{p_n\}_{n=1}^{\infty} \subseteq B_R(\vec{0})$

$$B_R(\vec{0}) \subseteq \underbrace{[-R,R] \times [-R,R] \times \cdots \times [-R,R]}_{\text{compact.}} = [-R,R]^k$$

$\{p_n\}_{n=1}^{\infty}$ is a sequence in $\underbrace{\text{compact}}_{\text{Heine Borel.}}$ $[-R,R]^k$

$\exists \{p_{n_k}\}_{k=1}^{\infty} \quad p_{n_k} \to p_0 \in [-R,R]^k \subseteq \mathbb{R}^k$.