Feb 26, 2020

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Connectedness

Def Let $A, B \subseteq (X, d)$

A & B are said to be separated if

$\bar{A} \cap B = \emptyset$

$A \cap \bar{B} = \emptyset$.

- $(\bar{A} \cap B = \emptyset \Rightarrow A \cap B = \emptyset)$

- $(0,1), (1,2)$ are separated since

$\overline{(0,1)} = [0,1]$

$[0,1] \cap (1,2) = \emptyset$

$(0,1) \cap [1,2] = \emptyset$.

- $(0,1), [1,2)$ are not separated

$\overline{(0,1)} \cap [1,2) = [0,1] \cap [1,2] = \{1\}$.

Def A subset $E \subseteq (X, d)$ is called connected if there doesnot exist any non-trivial separation of E:

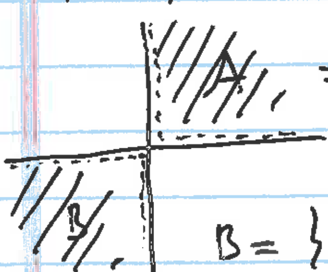
$\nexists\, A, B \subseteq X$ s.t. $A \neq \emptyset,\ B \neq \emptyset$

$A \cup B = E$

$\bar{A} \cap B = \emptyset$

$A \cap \bar{B} = \emptyset$

Ex $\{(x,y) \in \mathbb{R}^2 \mid xy > 0\} = E$



$A = \{(x,y) \mid xy > 0,\ x > 0,\ y > 0\} \neq \emptyset$

$A \cap \bar{B} = \emptyset,\ \bar{A} \cap B = \emptyset$

$B = \{(x,y) \mid xy > 0,\ x < 0,\ y < 0\} \neq \emptyset$

2.47 **Thm:** Let $E \subseteq \mathbb{R}^1$, $d(x,y) = |x-y|$.

E is connected $\iff \underbrace{\forall x,y \in E \left(\forall z \in \mathbb{R},\ x<z<y \Rightarrow z \in E\right)}_{E \text{ is an interval.}}$

Lemma: $E \subseteq F \Rightarrow \bar{E} \subseteq \bar{F}$ in any metric space

Also:
$\overline{F \cap G} \subseteq \bar{F} \cap \bar{G}$
proof:
$F \cap G \subseteq F$
$\overline{F \cap G} \subseteq \bar{F}$
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Proof $N_r(p) \cap (E - \{p\}) \subseteq N_r(p) \cap (F - \{p\})$
since $E \subseteq F$

If LHS $\neq \emptyset$, then RHS $\neq \emptyset$.

$\forall p \in E'$, $p \in F'$; $E' \subseteq F'$

$E \subseteq F$

$\bar{E} = E' \cup E \subseteq F' \cup F = \bar{F}$

Proof of Thm 2.47

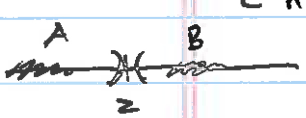
($\Rightarrow$:) Assume connected, Hypothesis (Proof by contradiction)

Suppose not $(\forall x,y \in E\ \forall z \in \mathbb{R},\ x<z<y \Rightarrow z \in E)$

$\exists x,y \in E\ \exists z \in \mathbb{R}\ x<z<y$ but $z \notin E$.

$E \cap (-\infty, z) = A$ $\qquad A \cap B = \emptyset$ since $(-\infty,z) \cap (z,\infty) = \emptyset$

$E \cap (z, \infty) = B$ $\qquad A \cup B = E$ since $z \notin E$



$x \in A$, $y \in B$, $A \neq \emptyset$, $B \neq \emptyset$.

$\bar{A} \cap B = ?$

$A = E \cap (-\infty, z)$, $\quad \bar{A} \subseteq (-\infty, z]$, since $A \subseteq (-\infty, z)$

$\bar{A} \cap B \subseteq (-\infty, z] \cap (z, \infty) = \emptyset$

$A \cap \bar{B} = \emptyset$. similarly

Which contradicts with the hypothesis.

$(\Leftarrow:)$ $E \subseteq \mathbb{R}$

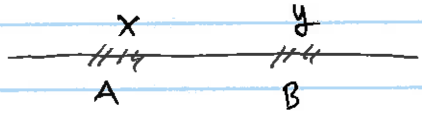
Assume $\underbrace{\forall x, y \in E \left(\forall z \in \mathbb{R},\ x < z < y \Rightarrow z \in E \right)}_{(*)}$

To prove E is connected.

Suppose E is not connected.

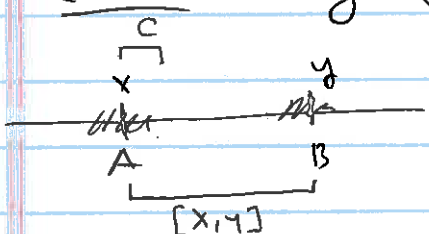
So $\exists$ a non-trivial seperation of E:

$$\begin{cases} E = A \cup B, & A \neq \emptyset,\ B \neq \emptyset \\ \bar{A} \cap B = \emptyset, & A \cap \bar{B} = \emptyset \end{cases}$$



$$\left.\begin{array}{l} \exists x \in A \subseteq E \\ \exists y \in B \subseteq E \end{array}\right\} \quad x \neq y.$$

Case 1 $x < y$ (Case $y < x$, same proof)



Let $C = A \cap [x, y]$

C is bounded above by y

$x \in C \neq \emptyset$

LUB property $\exists \sup C \in \mathbb{R}$, say $z = \sup C$.

$z \in \bar{C} \subseteq [x,y] \cap \bar{A}$

$z \in \bar{A}$

$z \notin B$ since $\bar{A} \cap B = \emptyset$

$x \leq z < y$ since $y \in B$.

Case A $z \notin A$

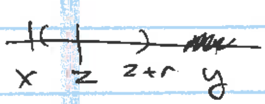
$z \notin A,\ z \notin B \implies z \notin E = A \cup B$.

Contradicts $(*)$

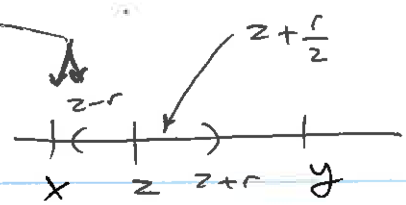
Case B $z \in A$

$z \notin \bar{B}$ since $A \cap \bar{B} = \emptyset$.

$\bar{B}$ closed $\implies \exists r > 0$ $\begin{cases} (z-r, z+r) \cap \bar{B} = \emptyset \\ \text{and} \\ r + z < y \end{cases}$ (since $z < y$)



order of x and $z-r$ is not relevant



$$x < z+\frac{r}{2} < y \quad , \quad z+\frac{r}{2} \in [x,y].$$

$$\sup C = z < z+\frac{r}{2} \notin A$$

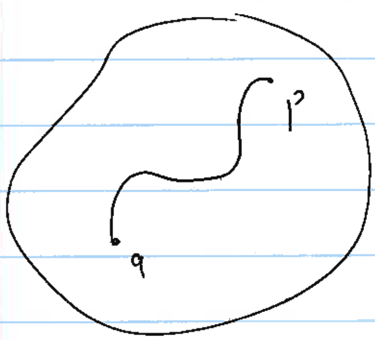
$C = A \cap [x,y]$

$$z+\frac{r}{2} \notin B \quad \text{since } (z-r, z+r) \cap \bar{B} = \phi.$$

$x, y \in E$, $z+\frac{r}{2}$ between x & y, but $z+\frac{r}{2} \notin E$
which contradicts $(*)$.

Not in the test:

<u>Path Connected</u> E is path connected if
$\forall p, q \in E$, $\exists$ curve $\gamma: [a,b] \to E$ $\to$ continuous
s.t. $\gamma(a) = p$, $\gamma(b) = q$

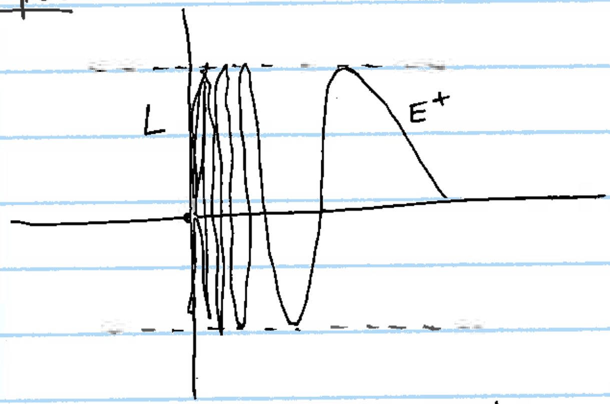


Path connected $\Longrightarrow$ connected.
$\overset{?}{\Longleftarrow}$ (not always)

in $\mathbb{R}^1$ connected $\Longrightarrow$ interval
$\Longrightarrow$ path connected

connected $\not\Rightarrow$ path connected even in $\mathbb{R}^2$

Example



$$E = \{(x, \sin \tfrac{1}{x}) \mid x > 0\} \cup \{(0,0)\}$$

$$E^+ = \{(x, \sin \tfrac{1}{x}) \mid x > 0\}$$

$$E = E^+ \cup \{(0,0)\}$$

$$\overline{E^+} = E^+ \cup L$$

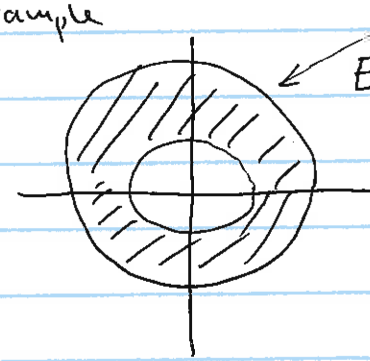
$$(0,0) \in L = \{(x,y) \mid x = 0, \ -1 \le y \le 1\}$$

E is connected

E is not path connected.

Remark: in $\mathbb{R}^k$, All k-cells are connected but connected sets are not always k-cells; or not products of intervals. Example E above.

Example



$E_0 = \{(x,y) \mid 1 \le x^2+y^2 \le 4\}$ is connected. and path connected