Feb 24, 2020

①

Thm 2.41 The following are equivalent for subsets $E \subseteq \mathbb{R}^n$, wrt standard metric

(a) E is closed & bounded
(b) E is compact
(c) Every infinite subset of E has a limit pt in E.

Proof a $\Leftrightarrow$ b Done (Feb 21, 2020)
b $\Rightarrow$ c Thm 2.37. (Feb 19, 2020)

To show: c $\Rightarrow$ a

Assume (c), to show (i) E is bounded
(ii) E is closed.

(i) Suppose E is not bounded.

$$\text{not}\left(\exists R>0,\ E \subseteq B_R(0)\right)$$
$$\forall R>0 \quad E \nsubseteq B_R(0)$$

Take $n \in \mathbb{N}$, $R = n$ $\exists z_n$ s.t. $z_n \in E$ $|z_n| \geq n$

Let $S = \{z_n \mid n \in \mathbb{N}\}$

Suppose S is finite, that is z_n repeats some values over & over.

$\{|z_n| \mid n \in \mathbb{N}\}$ is finite, hence there is a largest value, say $\ell = \max\{|z_n| \mid n \in \mathbb{N}\}$

By Archimedean Principle $\exists\ m \in \mathbb{N}$, $m > \ell$.

$$m \le |z_m| \le \ell < m \quad \times. \text{ contradiction}$$

So S is an infinite set.

By hypothesis (c), $S' \neq \phi$. $S' \cap E \neq \phi$.

$\exists\ q \in S' \cap E$.

$N_1(q)$ would contain infinitely many elements z_{n_k} of S. (prop 2.20) $k = 1, 2, 3, \ldots$

$$\forall z_{n_k} \in N_1(q) \cap S \qquad |z_{n_k} - q| < 1$$

$$\underset{\substack{\downarrow \\ \infty}}{n_k} \le |z_{n_k}| \le \underbrace{|q|}_{\text{fixed value}} + 1$$

Contradiction.

Hence E is bounded. (P10) for (ii)

To prove (ii) E is closed.

Suppose not, i.e. E is not closed: $E' \not\subseteq E$

$\exists p_0 \in E', \ p_0 \notin E$

$\forall n \in \mathbb{N} \quad N_{\frac{1}{n}}(p_0) \cap (E - \{p_0\}) \neq \emptyset.$

$\forall n \in \mathbb{N} \ \exists y_n \in E, \qquad 0 < |y_n - p_0| < \frac{1}{n}$

Let $S_0 = \{y_n \mid n \in \mathbb{N}\} \subseteq E$

Suppose S_0 is a finite set.

$\{|y_n - p_0| \mid n \in \mathbb{N}\}$ would be a finite set of positive real #'s

Let $c_0 = \min\{|y_n - p_0| \mid n \in \mathbb{N}\}$

$c_0 > 0$

$0 < c_0 \leq |y_n - p_0| < \frac{1}{n} \rightarrow n \to \infty$

fixed

Contradicts Archimedean Principle.

Hence S_0 is an infinite set.

Recall Hypothesis: (c)

Every infinite subset of E has an accumulation/limit pt in E.

S_0 is infinite

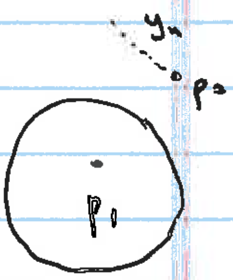
(c) $\implies$ $S_0' \cap E \neq \emptyset$.

actually $y_n \in S_0$

$p_0 \in S_0'$ since $\forall n \in \mathbb{N}\ \exists y_n \in E,\ 0 < |y_n - p_0| < \frac{1}{n}$

Suppose S_0' has other points in it, say p_1

Is it possible to have $p_1 \in S_0'$, $p_1 \in E$, $p_1 \neq p_0$, $p_0 \notin E$?



$$|p_1 - y_n| \geq |p_1 - p_0| - |p_0 - y_n|$$

$$\geq |p_1 - p_0| - \frac{1}{n} \geq \underbrace{|p_1 - p_0| - \frac{|p_0 - p_1|}{2}}_{= \frac{|p_0 - p_1|}{2}}$$

If I choose $n > \frac{2}{|p_0 - p_1|}$

$N_{\frac{|p_1 - p_0|}{2}}(p_1)$ contains at most $\underbrace{\text{finitely many}}_{\# \leq \frac{2}{|p_0 - p_1|}}$ of y_n.

$S_0 = \{y_n | n \in \mathbb{N}\} \implies p_1 \notin S_0'$ (prop 2.20)

$S_0' = \{p_0\}$, $p_0 \notin E$

$\implies S_0' \cap E = \emptyset$. which contradicts (c).

Hence: E is closed

Big picture:

(a) E is closed and bounded
(b) E is compact
(c) Every infinite subset of E has a limit pt in E

- In $\mathbb{R}^k$:

$$\begin{tikzcd} (a) \arrow[r, Leftrightarrow] \arrow[dr, Leftrightarrow] & (b) \arrow[d, Rightarrow] \\ & (c) \end{tikzcd}$$

- In Any Metric Space

$$\begin{tikzcd} (a) & (b) \arrow[l, Rightarrow] \arrow[d, Leftrightarrow] \\ & (c) \arrow[ul, Rightarrow] \end{tikzcd}$$

(a) $\not\Rightarrow$ (b)

1) $(\mathbb{Q}, \|\cdot\|)$ $\qquad [0,1] \cap \mathbb{Q}$ is closed in $\mathbb{Q}$
bdd in $\mathbb{Q}$
but $[0,1]\cap\mathbb{Q}$ is not compact.

2) $\{ f : [a,b] \to \mathbb{R} \mid f \text{ continuous} \}$

$$d_\infty(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$$

(Chap 7) closed & bounded $\not\Rightarrow$ compactness.

- Topological spaces

$$\begin{tikzcd} (b) \arrow[r, Rightarrow, "\text{Hausdorff}"] \arrow[d, Rightarrow, "\times"'] & \text{closed} \ (\text{bounded not defined}) \\ (c) & \end{tikzcd}$$

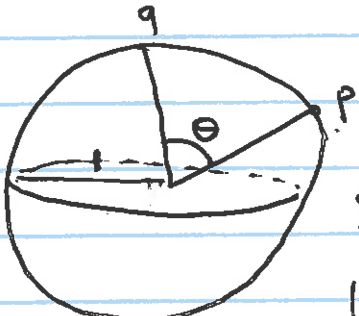
However Let M, A be given fixed numbers.

$$\left\{ f: [a,b] \to \mathbb{R} \;\middle|\; \begin{array}{l} \forall x,y \quad |f(x)-f(y)| \le M|x-y| \\ \forall x \quad |f(x)| \le A \end{array} \right\}$$

Compact wrt $d_\infty(f,g) = \sup_{x\in[a,b]} |f(x)-g(x)|$.

Other **Examples** of metric spaces:

(1)



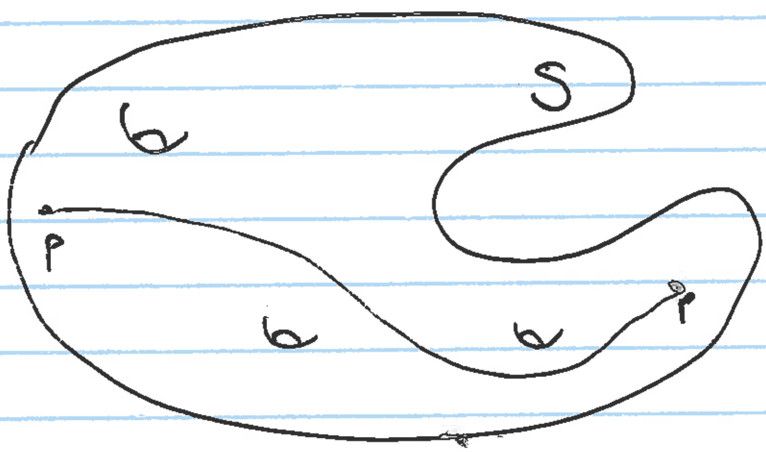
unit sphere

$$S^2 = \{(x,y,z) \mid x^2+y^2+z^2=1\} \subseteq \mathbb{R}^2$$

$$d_{S^2}(p,q) = \Theta = \cos^{-1}(p\cdot q)$$

(S^2, d_{S^2}) compact metric space

(2)



(no punctures!)

compact metric space

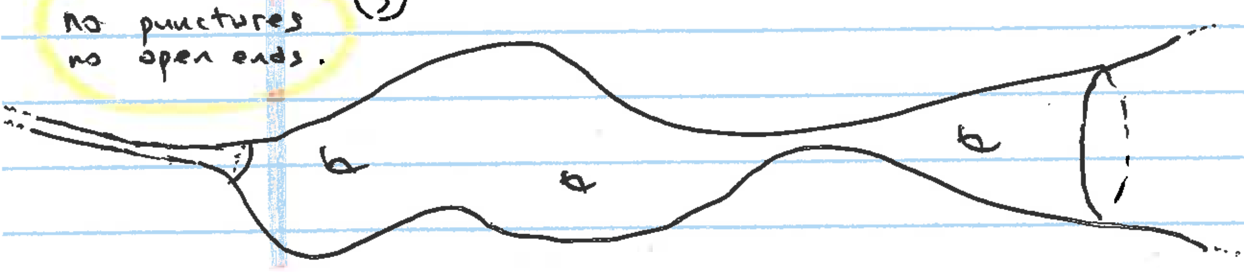
A donut with 3 holes

$$d(p,q) = \inf \left\{ \ell(\gamma) \;\middle|\; \gamma \text{ is a piecewise } C^1 \text{ curve on } S \text{ connecting } p \text{ to } q \right\}$$

ℓ ↗ length

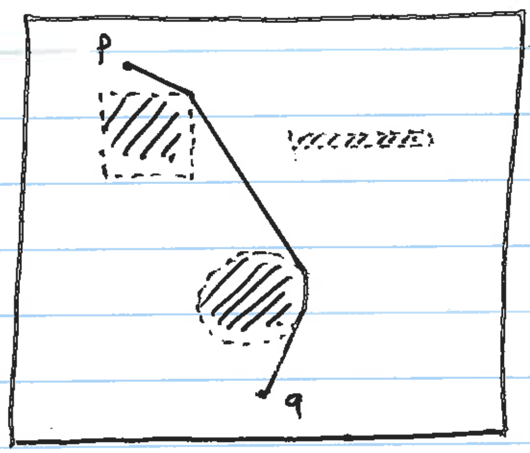
(3)

no punctures
no open ends.



Complete metric space

4



A yard/garden with some obstructions

$d^L(p,q)$ = infimum of lengths of curves from p to q, in the yard avoiding obstructions

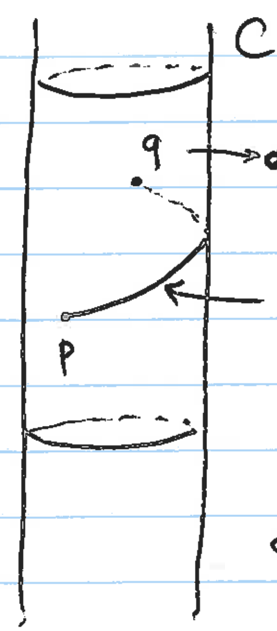
This is a length-space. A compact metric space with respect to d^L.

(yard-obstructions need to be closed & bounded and path connected wrt standard metric of $\mathbb{R}^2$: $d(p,q) = \|p-q\|$.)

less

5 Infinite cylinder

$\{(x,y,z) \mid x^2+y^2=1\} = C$



on the back.

shortest curve on C from p to q is a part of a helix.

$d(p,q)$ = length of this curve (geodesic)

This is a complete metric