Feb 21, 2020

1

Compactness in $\mathbb{R}^k$.

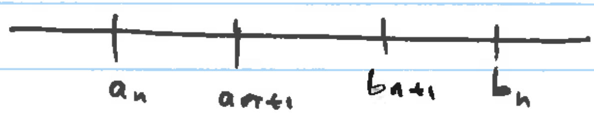
Prop (2.38) Let $[a_n, b_n]$ be a sequence of non-empty closed intervals in $\mathbb{R}$ (standard metric) s.t.

$$\emptyset \neq I_{n+1} \subseteq I_n \quad \forall n \in \mathbb{N}.$$

Then
$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

Proof: $I_n = [a_n, b_n]$. , $a_n \leq b_n$.

$$[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$$



$\forall n, m \in \mathbb{N}$

$$a_n \leq a_{n+m} \leq b_{n+m} \leq b_m$$

Let $E = \{a_n \mid n \in \mathbb{N}\} \neq \emptyset$.

E is bounded above by each of b_i

LUB prop $\Longrightarrow$ $\exists A = \sup E \in \mathbb{R}$.

$\forall n \quad a_n \leq A \qquad (\sup E = A)$

$\forall m \qquad A \leq b_m$

$\uparrow$ lub $\quad \uparrow$ an upper bd for E

$A \in [a_n, b_n] \; \forall n$.

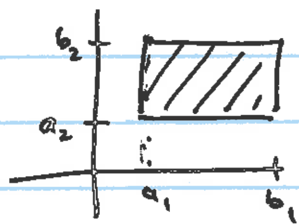
$$A \in \bigcap_{n=1}^{\infty} [a_n, b_n] \neq \emptyset.$$

Defn k-cells in $\mathbb{R}^k$:

$$\left\{ (x_1, x_2, \dots x_k) \in \mathbb{R}^k \;\middle|\; \begin{array}{c} a_1 \leq x_1 \leq b_1 \\ a_2 \leq x_2 \leq b_2 \\ \vdots \\ a_k \leq x_k \leq b_k \end{array} \right\}$$

for some given $a_i, b_j \in \mathbb{R}$.

Ex



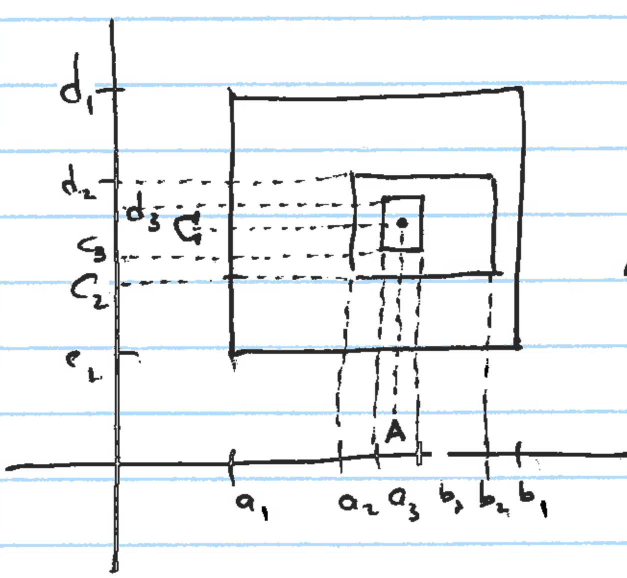
in $\mathbb{R}^2$

2.39 **Prop** Let $k \in \mathbb{N}$, Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of k-cells in $\mathbb{R}^k$ s.t.

$$\phi \neq I_{n+1} \subseteq I_n, \; \forall n \in \mathbb{N}$$

Then $\bigcap_{n=1}^{\infty} I_n \neq \phi$.

Proof $n=2$ case in class, $n \geq 2$ in the book
HW to read



$$I_n = [a_n, b_n] \times [c_n, d_n]$$

$$[a_n, b_n] \supseteq [a_{n+1}, b_{n+1}]$$

$$[c_n, d_n] \supseteq [c_{n+1}, d_{n+1}]$$

$$A \in \bigcap_{n=1}^{\infty} [a_n, b_n] \neq \phi$$

$$C \in \bigcap_{n=1}^{\infty} [c_n, d_n] \neq \phi$$

$$(A, C) \in \bigcap_{n=1}^{\infty} I_n \neq \phi.$$

***** **Thm (2.40)** Every k-cell in $\mathbb{R}^k$ is compact.

HEINE-BOREL

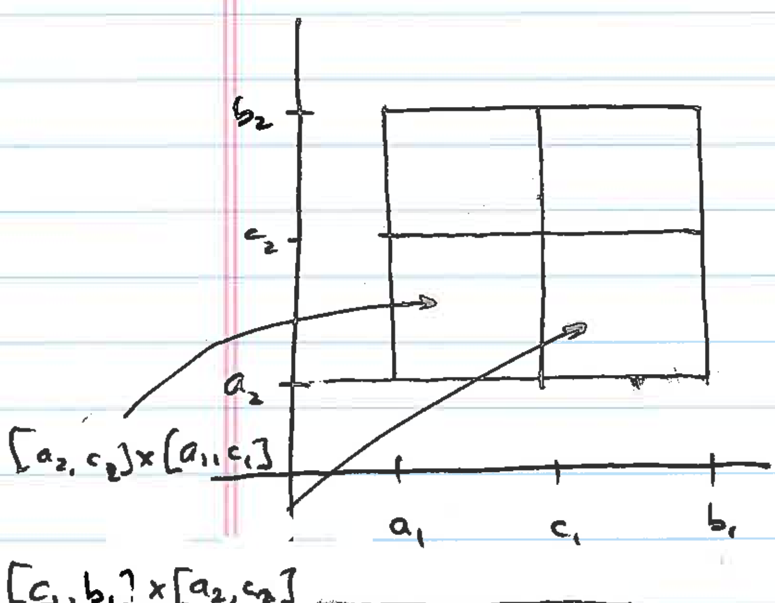
Corollary: Let $K \subseteq \mathbb{R}^k$. Then

K is compact $\iff$ K is closed and bounded.

Proof of Thm 2.40

If $I = \emptyset$, it is compact. WLOG $I \neq \emptyset$.

Let $I = \{(x_1, x_2, \ldots, x_k) \mid a_i \leq x_i \leq b_i$ for $i = 1, 2, \ldots k\}$
$= [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k]$.



Let $\delta = \left(\sum_{i=1}^{k} (b_i - a_i)^2 \right)^{\frac{1}{2}}$

$\delta = \mathrm{diam}(I)$

$\forall p, q \in I, \ \|p - q\| \leq \delta$

(*) Suppose I is not compact, and there is an open cover $\mathcal{G} = \{G_\alpha \mid \alpha \in \Lambda\}$ of I s.t. no finite subcollection of $\mathcal{G}$ can cover I.

open

Let $c_j = \dfrac{a_j + b_j}{2}$. $\forall j = 1, \ldots k$

Divide I into 2^k subcells Q_i by

taking products of $[a_j, c_j]$ or $[c_j, b_j]$ as j varies $1, 2, \dots k$.

e.g. $[a_1, c_1] \times [c_2, b_2] \times [c_3, b_3] \times [a_4, c_4] \times \cdots$

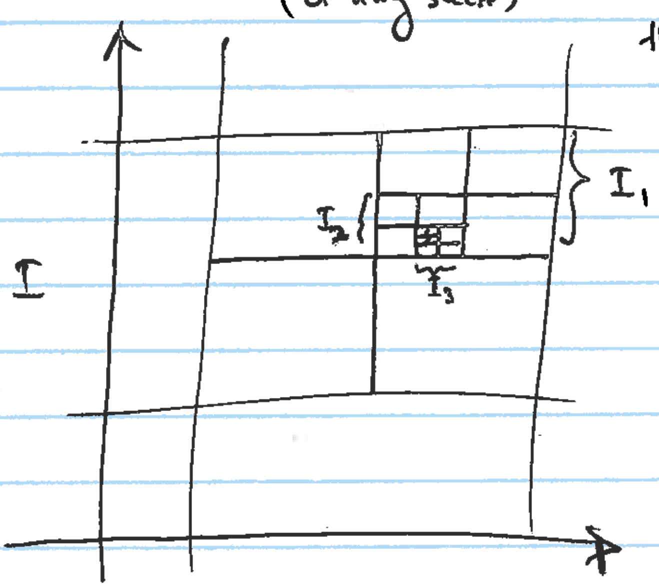
2 possibilities each, total of 2^k possibilities.

If we can cover each of these Q_i $(1 \le i \le 2^k)$ with finitely many of G_α's, then we would be able to cover all of I with finitely many G_α's.

But, I cannot be covered by using finitely many G_α's, so at least one of Q_i cannot be covered with finitely many G_α's.

Call that (or any such) subcell I_1. $\operatorname{diam} I_1 = \frac{\delta}{2}$.

that cannot be covered by finitely many G_α.



Now subdivide I_1 into 2^k equal size subcells by dividing each edge into 2 equal pieces along each axis direction. One of the 2^k (new) subcells of I_1 cannot be covered by using finitely many G_α's. Call that cell (or any such cell) I_2.

$$\underset{\text{diam } \frac{\delta}{4}}{I_2} \subseteq \underset{\text{diam} = \frac{\delta}{2}}{I_1} \subseteq \underset{\text{diam} = \delta}{I}$$

Inductively, we proceed by subdividing I_{n-1} into 2^k equal size subcells, and choosing one which cannot be covered by finitely many G_α's, and calling it I_n.

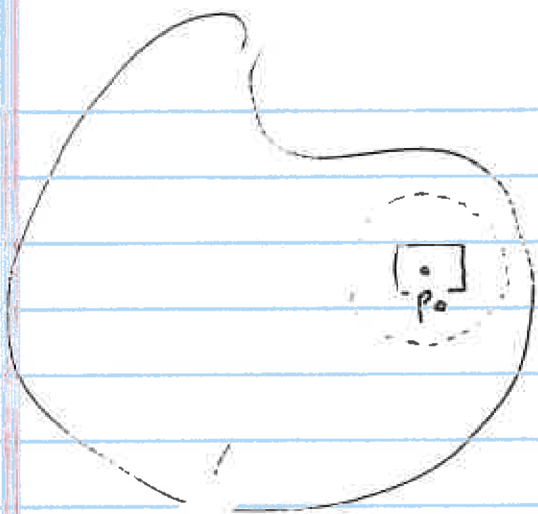
- $I \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1}$
- $\text{diam } I_n = \dfrac{\delta}{2^n}$
- Each I_n cannot be covered by finitely many G_α's

$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. Let $p_0 \in \bigcap_{n=1}^{\infty} I_n$.

$$p_0 \in \underset{\forall n}{I_n} \subseteq I \subseteq \bigcup_{\alpha \in \Lambda} G_\alpha.$$

$\exists\, G_{\alpha_0}$ s.t. $p_0 \in G_{\alpha_0}$ which is open.

$$\mathscr{C} = \{ \underset{\text{open}}{G_\alpha} \mid \alpha \in \Lambda \}$$



G_{α_0} open.

$\exists r > 0$ s.t. $N_r(p_0) \subseteq G_{\alpha_0}$.

$\exists n_0 \in \mathbb{N}$ s.t. $\frac{\delta}{2^{n_0}} < r$.

$\operatorname{diam} I_{n_0} = \frac{\delta}{2^{n_0}}$

$p_0 \in I_{n_0}$ since $p_0 \in \bigcap_{n=1}^{\infty} I_n$.

$\forall q \in I_{n_0}$, $\|p_0 - q\| \leq \operatorname{diam} I_{n_0} = \frac{\delta}{2^{n_0}} < r$. $\forall q \in I_n$, $q \in N_r(p_0)$

$I_{n_0} \subseteq N_r(p_0) \subseteq G_{\alpha_0}$

I_{n_0} is covered by one G_{α_0}.

Contradiction, since in the construction we took each I_n to be not coverable by finitely many G_α's. Hence I is compact. End of 2.40. #

Corollary Let $K \subseteq \mathbb{R}^k$. Then:

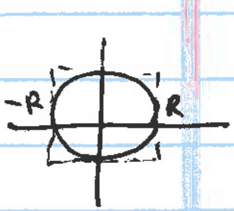
K is compact $\Longleftrightarrow$ K is closed & bounded.

Proof "$\Longrightarrow$" done before. (on Feb 19, 2020)

"$\Longleftarrow$" K bounded (given). $\exists R > 0$ $B_R(0) \supseteq K$.

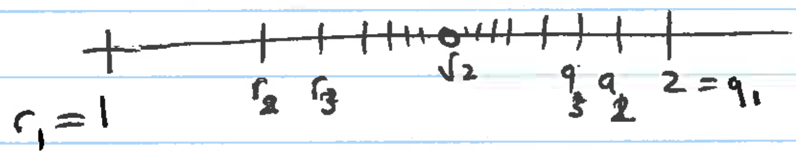
$$K \subseteq B_R(0) \subseteq \underbrace{[-R, R] \times [-R, R] \times \cdots \times [-R, R]}_{\text{compact}}.$$

closed (given)



$\Longrightarrow K$ compact (Thm 2.35)

Ex $S=[0,2]\cap\mathbb{Q}$, $d(p,q)=\|p-q\|$.



$$\bigcap_{n=1}^{\infty}[r_n,q_n]\cap\mathbb{Q}=\emptyset,$$ if we choose r_n, q_n carefully.

By using density of rationals:

Choose $r_1=1$, $q_1=2$ $\quad 1,2\in S$.

Inductively if we have

- $1=r_1<r_2<\dots<r_n<\sqrt{2}<q_n<q_{n-1}<\dots<q_1=2$
- $\forall i=1,2,\dots n$ $\quad r_i\in S$, $q_i\in S$

$|r_i-\sqrt{2}|<\frac{1}{i}$

$|q_i-\sqrt{2}|<\frac{1}{i}$

Choose $r_{n+1}\in S$ s.t. $\max(r_n,\sqrt{2}-\frac{1}{n+1})<r_{n+1}<\sqrt{2}$

$q_{n+1}\in S$ s.t. $\min(q_n,\sqrt{2}+\frac{1}{n+1})>q_{n+1}>\sqrt{2}$

$$\emptyset\neq[r_{n+1},q_{n+1}]\cap\mathbb{Q}\subseteq[r_n,q_n]\cap\mathbb{Q}$$

$$\bigcap_{l=1}^{m}[r_n,q_n]\cap\mathbb{Q}\neq\emptyset$$

$$\bigcap_{l=1}^{\infty}[r_n,q_n]\cap\mathbb{Q}=\emptyset.$$ since $|r_n-q_n|<\frac{2}{n}$ & $\sqrt{2}\notin S$.