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# Chap II COMPACTNESS (Revisited)

**Defn** Let $(X,d)$ be a metric space, $E \subseteq X$.
A collection $\{G_\alpha \mid \alpha \in \Lambda,\ G_\alpha \text{ is an open set in } X\}$ of open sets is called a cover of $E$ if

$$E \subseteq \bigcup_{\alpha \in \Lambda} G_\alpha$$

**Defn** A subset $K$ of $(X,d)$ is called compact, if every open cover of $K$ has a finite subcover.

i.e.

$$\left( K \subseteq \bigcup_{\alpha \in \Lambda} G_\alpha,\ \text{all } G_\alpha \text{ open} \right)$$

$$\Rightarrow \left( \exists\, \alpha_1, \alpha_2, \ldots \alpha_\ell \in \Lambda \text{ s.t. } \bigcup_{i=1}^{\ell} G_{\alpha_i} \supseteq K \right)$$

**Obs** $\emptyset$ compact
Every finite set is compact

$\mathbb{R}$ is not compact : Why?

$$\bigcup_{n=1}^{\infty} \underbrace{(-n, n)}_{\text{open intervals}} = \mathbb{R}$$

for $\displaystyle\bigcup_{i=1}^{\ell} (-n_i, n_i) = (-R, R) \neq \mathbb{R}$.
Let $R = \max(n_1, n_2, \ldots n_\ell)$

$\{(-n,n) \mid n \in \mathbb{N}\}$ has no finite subcover for $\mathbb{R}$.

**Thm 2.34** Compact subsets of metric spaces are closed.

**Remark:** Converse is false: $\mathbb{R}$ is closed in $\mathbb{R}$, not compact

**Proof:** $K \subseteq X$, $K$ compact. (Given)

"Want to show" $\longrightarrow$ WTS $K$ closed

Suffices to show $K^c = X - K$ is open.


Choose any $p \in K^c$, fix it.

Let $q$ be an arbitrary pt of $K$

$$d(p,q) > 0$$

Define:

$$q \in W_q = N_{\frac{1}{2}d(p,q)}(q)$$

$$V_q = N_{\frac{1}{2}d(p,q)}(p)$$

open

$\forall q \in K \quad W_q \cap V_q = \emptyset$

$$\bigcup_{q \in K} W_q \supseteq K \quad \text{since } q \in W_q.$$

open

$K$ compact $\Rightarrow \exists$ finite subcover:

$$\bigcup_{i=1}^{\ell} W_{q_i} \supseteq K. \quad \text{for some } q_1, q_2, \dots q_\ell \in K.$$

Let $V = \bigcap_{i=1}^{\ell} V_{q_i}$ is open since $\left\{ \begin{array}{l} \text{intersection of} \\ \text{finitely many} \\ \text{open sets} \end{array} \right.$

$V \subseteq V_{q_i}$ for all $i = 1, \dots, \ell$

$V \cap K \subseteq V_{q_i} \cap K.$

$V \cap W_{q_i} \subseteq V_{q_i} \cap W_{q_i} = \emptyset.$

$V \cap \left( \bigcup_{i=1}^{\ell} W_{q_i} \right) = \bigcup_{i=1}^{\ell} (V \cap W_{q_i}) = \emptyset.$

$V \cap K \subseteq V \cap \left( \bigcup_{i=1}^{\ell} W_{q_i} \right) = \emptyset.$

$V$ open, neighborhood of $p$.
$p \in V$
$V \subseteq K^c$ since $V \cap K = \emptyset$.

$\Rightarrow$ $K^c$ is open, since $p$ was arbitrary
$\Rightarrow$ $K$ is closed.

<u>Thm 2.35</u> In a metric space, closed subsets of compact sets are compact.

i.e. $F \subseteq K \subseteq X$ metric space

$F$ closed, $K$ compact $\Rightarrow$ $F$ compact

<u>Proof:</u> Let $\mathscr{G} = \{G_\alpha \mid \alpha \in \Lambda\}$ be an open cover of $F$.

$F$ closed $\Rightarrow$ $F^c$ is open.

$\mathscr{G}_0 = \mathscr{G} \cup \{F^c\}$ covers all of $X$
hence it covers $K$.

$K$ compact $\Rightarrow$ $\exists$ a finite subcover $\mathscr{G}_1$ of $\mathscr{G}_0$,
$\mathscr{G}_1$ which covers $K$.

$\mathscr{G}_1$ covers $F$ since $F \subseteq K$.

$F^c$ covers no part of $F$, so

$\underset{\text{finite}}{\mathscr{G}_1} - \underbrace{\{F^c\}}_{\text{finite subcover}}$ still cover $F$.

$F$ compact, since all open covers of $K$ has a finite subcover.

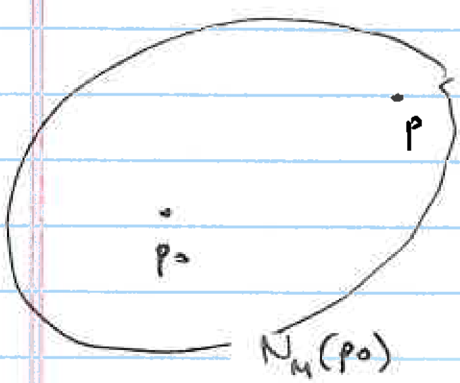
<u>Cor.</u> $F$ closed, $K$ compact $\Rightarrow$ $F \cap K$ compact

<u>Thm:</u> Every compact set $K$ in a metric space is bounded.

<u>Proof</u> Choose any pt $p_0 \in K \subseteq (X, d)$
(If $K = \phi$, nothing to prove!)

$\bigcup_{n=1}^{\infty} N_n(p_0)$ covers all $X$, since:

$\forall p \in X$, $d(p, p_0) < \infty$
$\Longrightarrow \exists M \in \mathbb{N}$ s.t. $d(p, p_0) < M$
$p \in N_M(p_0)$.




$\forall p \in X$, $p \in \bigcup_{n=1}^{\infty} \underbrace{N_n(p_0)}_{\text{open}}$.

$K \subseteq \underbrace{\bigcup_{n=1}^{\infty} N_n(p_0)}_{\text{open cover for } K}$.

$\exists$ finite subcover $K \subseteq \bigcup_{i=1}^{\ell} N_{n_i}(p_0)$
for some $n_1, n_2, \dots n_\ell \in \mathbb{N}$

Let $R = \max(n_1, n_2, \dots n_\ell)$

$$K \subseteq \bigcup_{i=1}^{\ell} N_{n_i}(p_0) = N_R(p_0)$$

$K$ is bounded since $\exists R > 0$ s.t. $K \subseteq N_R(p_0)$.

**Thm 2.37** Let $E$ be an infinite subset of a compact set $K$, $K \subseteq (X, d)$ a metric space. Then $E$ must have at least one limit pt in $K$: $E' \cap K \neq \emptyset$.

**Proof:** Suppose not: $E' \cap K = \emptyset$ ($\sim$ conclusion)
$E$ infinite, $K$ compact (hypothesis)

$\forall q \in K$, $q \notin E'$.
not ( $\forall r > 0$ $N_r(q) \cap (E - \{q\}) \neq \emptyset$ )

$\exists r_q > 0$ $\underbrace{N_{r_q}(q)}_{\text{Call } V_q,\ q \in V_q} \cap (E - \{q\}) = \emptyset$.

adding at most one pt.

$V_q \cap E \subseteq \{q\}$

$K \subseteq \bigcup_{q \in K} V_q$, $K$ compact, $V_q$ open

$\exists\, q_1, q_2, \ldots q_\ell$ s.t. $K \subseteq \bigcup_{i=1}^{\ell} V_{q_i}$

$E \subseteq K$

$$\underset{\text{infinite set}}{E} = E \cap K \subseteq E \cap \bigcup_{i=1}^{\ell} V_{q_i} = \bigcup_{i=1}^{\ell} (E \cap V_{q_i}) \subseteq \underset{\text{finite}}{\{q_1, q_2, \ldots q_\ell\}}$$

$\subseteq$

Contradiction. Conclusion $E' \cap K \neq \emptyset$.

Overview:

1. In Every metric space:

$$K \text{ is compact} \implies K \text{ is closed \& bounded.}$$

2. Converse may not be true. (Many Examples)

Heine-Borel Thm: 3. In $\mathbb{R}^n$ $\left( K \text{ compact} \iff K \text{ is closed \& bounded} \right)$