1. Provide the definitions of the following concepts in arbitrary metric spaces (X, d) unless it is specified otherwise. The definitions you give need to be the same (or have the same meaning) as of those definitions given in the textbook or in class, without using a theorem or proposition which require a proof.

a. A sequence $\{p_n\}$ converging to a point p

If $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, $n \geq N \Rightarrow d_{\underline{X}}(p_n, p) < \varepsilon$.

b. A convergent series $\sum_{n=1}^{\infty} c_n$ in $\mathbb{R}$

Let $A_n = \sum_{k=1}^{n} c_n$. If $\{A_n\}$ is a convergent sequence, then the series $\sum_{n=1}^{\infty} c_n$ converges. If $\lim_{n\to\infty} A_n = s$, then $\sum_{n=1}^{\infty} c_n = s$.

c. An absolutely convergent series $\sum_{n=1}^{\infty} a_n$ in $\mathbb{R}$

$\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.

d. A continuous function from one metric space into another

A function $f: (\underline{X}, d_X) \to (Y, d_Y)$ is called continuous if $\forall p \in \underline{X} \ \forall \varepsilon > 0 \ \exists \delta > 0 \ \ni. \ \forall q \in \underline{X} \ \ d_{\underline{X}}(p, q) < \delta \Rightarrow d_Y(f(p), f(q)) < \varepsilon$

e. The limit of a function (from one metric space into another) at given point

For a function $f: (\underline{X}, d_X) \to (Y, d_Y)$, and $p \in \underline{X}$, the limit of f at p is $q \in Y$ if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall x \in \underline{X}$, $d_{\underline{X}}(x, p) < \delta \Rightarrow d_Y(f(x), q) < \varepsilon$. Notation $\lim_{x\to p} f(x) = q$.

2. Let $f : (X, d_X) \to (Y, d_Y)$ and $g : (Y, d_Y) \to (Z, d_Z)$ be uniformly continuous functions on metric spaces. Prove that $h = g \circ f$ is uniformly continuous on X.

Defn of uniform continuity:

$\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall x_1, x_2 \in X$ $d_X(x_1, x_2) < \delta \Rightarrow$
$d_Y(f(x_1), f(x_2)) < \varepsilon$

Proof of 2:

Let $\varepsilon > 0$ be given. By using unif. continuity of g

$\exists \eta > 0$ s.t. $\forall y_1, y_2 \in Y$

(*) $d_Y(y_1, y_2) < \eta \Rightarrow d_Z(g(y_1), g(y_2)) < \varepsilon.$

For the η we have, by using unif. continuity of f

$\exists \delta > 0$ s.t. $\forall x_1, x_2 \in X$

$d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \eta.$

In summary for the given $\varepsilon > 0$, we find $\eta > 0$ first, then find $\delta > 0$ from η.

$\forall x_1, x_2 \in X$ $\quad d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \eta$

by taking $y_1 = f(x_1)$ & $y_2 = f(x_2)$ in (*) above

$d_Y(f(x_1), f(x_2)) < \delta \Rightarrow d_Z(g(f(x_1)), g(f(x_2))) < \varepsilon.$

Hence $\forall \varepsilon > 0\ \exists \delta > 0$ s.t.

$\forall x_1, x_2 \in X$ $\quad d_X(x_1, x_2) < \delta \Rightarrow d_Z(g(f(x_1)), g(f(x_2))) < \varepsilon.$

$g \circ f$ is uniformly continuous by defn.

3. If $\sum_{n=0}^{\infty} a_n$ converges and $\{b_n\}$ is a bounded and decreasing sequence,
prove that $\sum_{n=0}^{\infty} a_n b_n$ converges. You must provide a precise statement of every theorem that you use in your solution and we have proven in class or in the textbook.

(*) Recall Thm (3.42 p 70)

If (i) Partial sums $A_n = \sum_{k=0}^{n} a_k$ form a bounded sequence, and

(ii) $b_0 \geq b_1 \geq \cdots \geq b_n \geq b_{n+1} \geq \cdots$, and

(iii) $\lim_{n\to\infty} b_n = 0$

then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Proof of #3. $\sum_{n=0}^{\infty} a_n$ converges

$A_n = \sum_{k=0}^{n} a_k$ converges by def.

A_n is bounded. (convergent $\Rightarrow$ bdd. Thm 3.2 c)

$\{b_n\}$ is a bounded and decreasing sequence, hence

$\lim_{n\to\infty} b_n = b$ exists. (Thm. bdd + monotone $\Rightarrow$ converg. Thm. 3.14)

$\lim_{n\to\infty} b_n - b = 0$ (Thm 3.3)

Let $b_n' = b_n - b$. $b_n \geq b_{n+1} \Rightarrow b_n' \geq b_{n+1}'$.

$\{A_n\}$, $\{b_n'\}$ satisfy Thm(*) above

Hence $\sum_{n=1}^{\infty} a_n b_n'$ converges.

$$\underbrace{\sum_{n=1}^{\infty} a_n b_n'}_{\text{convergent}} + b \underbrace{\sum_{n=1}^{\infty} a_n}_{\text{convergent}} \Rightarrow \underbrace{\sum_{n=1}^{\infty} a_n b_n' + b a_n}_{\text{converges}} = \sum_{n=1}^{\infty} a_n (\underbrace{b_n' + b}_{b_n}) = \sum_{n=1}^{\infty} a_n b_n$$

Thm in the book p. 54.

4. Prove that every compact metric space is complete, that is, every Cauchy sequence in a compact metric space is convergent. You may assume *diam*(*E*) = *diam*($\overline{E}$) without proof.

You are asked to prove Theorem 3.11(b) which we proved in class also. You may use any axiom or proposition/theorem proven earlier, but you can neither use nor refer to Theorem 3.11(b) and the theorem we proved in class or their consequences. Simply, saying "This follows Theorem 3.11(b) and the theorems we proved in class" will not earn any credit. You are expected to provide proofs.

Let $\{p_n\}_{n=1}^{\infty}$ be a Cauchy sequence in (X, d) where X is compact. To prove $\exists p \in X$ s.t. $p_n \to p$.

$\forall N \in \mathbb{N}$, Let $E_N = \{p_n \mid n \geq N\}$. $\phi \neq E_{N+1} \subseteq E_N \subseteq \overline{E_N}$ which is closed.

$\phi \neq \overline{E_{N+1}} \subseteq \overline{E_N}$ by prop 2.27c

$\forall N$, $\overline{E_N}$ is a closed subset of compact X, so $\overline{E_N}$ is compact prop 2.35.

$\bigcap_{N=1}^{\infty} \overline{E_N} \neq \phi$ by corollary of Thm 2.36

Claim $\bigcap_{N=1}^{\infty} \overline{E_N} = \{p\}$ is a single pt. Suppose $\exists p, q \in \bigcap_{N=1}^{\infty} \overline{E_N}$ & $p \neq q$.

$\{p_n\}$ Cauchy $\Rightarrow$ $\forall \varepsilon > 0$ $\exists N_1$ $\forall n, m \geq N_1$ $d_X(p_n, p_m) < \varepsilon$.

$\forall \varepsilon > 0$ $\exists N_1$ $\forall n \geq N_1$ diam $E_n < \varepsilon$

$\forall \varepsilon > 0$ $\exists N_1$ $\forall n \geq N_1$ diam $\overline{E_n} < \varepsilon$

(diam E = diam $\overline{E}$) given

$\lim_{n \to \infty}$ diam $\overline{E_n} = 0$.

$0 < d(p, q) =$ diam $\{p, q\} \leq$ diam $\overline{E_n}$ $\forall n$

contradiction.

So we have $\bigcap_{N=1}^{\infty} \overline{E_N} = \{p\}$ for some p, i.e. $p \in \overline{E_N}$ $\forall N$.

Let $\varepsilon > 0$ be given: $\exists N_1$ s.t. diam E_{N_1} = diam $\overline{E_{N_1}} < \varepsilon$, and $p_n \in \overline{E_{N_1}}$ $\forall n \geq N_1$.

Hence $\forall n \geq N_1$, $d(p, p_n) < \varepsilon$

$\Rightarrow \lim_{n \to \infty} p_n = p$

$\{p_n\}$ converges.

(X, d) is complete.

Thm in the book p 93

5. Prove that if $f : (X, d_X) \to (Y, d_Y)$ is a continuous function and E is a connected subset of X, then $f(E)$ is connected in Y.

You are asked to prove Theorem 4.22 which we proved in class also. You may use any axiom or proposition/theorem proven earlier, but you can neither use nor refer to Theorem 4.22 and the theorem we proved in class or their consequences. Simply, saying "This follows Theorem 4.22 and the theorems we proved in class" will not earn any credit. You are expected to provide proofs.

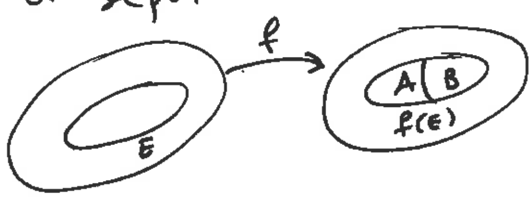
Defn A set C is connected if $\exists$ no separation of C i.e.

$\exists$ no $A, B \subseteq X$, $A \neq \phi$, $B \neq \phi$, and

$A \cup B = C$ and

$\bar{A} \cap B = A \cap \bar{B} = \phi$.

To prove $f(E)$ is connected if E is connected, we will start with a separation of $f(E)$ to obtain a separation of E.



Suppose $\exists A, B \subseteq Y$ s.t. $A \neq \phi$, $B \neq \phi$

$A \cup B = f(E)$

$\bar{A} \cap B = A \cap \bar{B} = \phi$.

Let $G = E \cap f^{-1}(A)$

$H = E \cap f^{-1}(B)$.

To show (i) $G \neq \phi$, $H \neq \phi$

(ii) $G \cup H = E$

(iii) $\bar{G} \cap H = G \cap \bar{H} = \phi$.

(i) $A \neq \phi$, $A \subseteq f(E)$

$\exists a \in A$, $a \in f(E)$. $a = f(c)$ for some $c \in E$.

$c \in f^{-1}(A)$ since $f(c) = a \in A$.

$c \in E \cap f^{-1}(A) = G \neq \phi$.

Similarly for $H \neq \phi$.

(ii) $G = E \cap f^{-1}(A) \subseteq E$

$H = E \cap f^{-1}(B) \subseteq E$

$G \cup H \subseteq E$. (1)

Let $e \in E$ be arbitrary

$f(e) \in f(E) = A \cup B$.

$f(e) \in A \Rightarrow e \in f^{-1}(A) \cap E = G$.

$f(e) \in B \Rightarrow e \in f^{-1}(B) \cap E = H$

$E \subseteq G \cup H$. (2) $E = G \cup H$ by (1) & (2)

(iii) $A \subseteq \bar{A}$

$G = f^{-1}(A) \cap E \subseteq f^{-1}(A) \subseteq \underbrace{f^{-1}(\bar{A})}_{\text{closed by } f \text{ continuous}}$

$\bar{G} \subseteq f^{-1}(\bar{A})$ (prop 2.27c)

$\bar{G} \cap H \subseteq f^{-1}(\bar{A}) \cap f^{-1}(B)$

$= f^{-1}(\underbrace{\bar{A} \cap B}_{\phi}) = \phi$.

$\bar{G} \cap H = \phi$.

$G \cap \bar{H} = \phi$ similarly.

6. TRUE OR FALSE CIRCLE YOUR ANSWERS.
NO PARTIAL CREDITS. YOU ARE NOT EXPECTED TO SHOW WORK.
Correct answers are +4 points each,
wrong answers are -1 point each,
ambiguous answers are -2 points each, and
no answers are 0 point each.
Total of problem 5 will be added to your total grade only if it is positive.
HINT: Read very carefully.

TRUE FALSE **a.** Given sequences $\{x_n\}$ and $\{y_n\}$ in $\mathbb{R} - \{0\}$, $c_n = (x_n, y_n)/\|(x_n, y_n)\|$ has a convergent subsequence in $\mathbb{R}^2$ with respect to the standard metric.

$\|c_n\| = 1$ So $\{c_n\}$ is a bounded sequence in $\mathbb{R}^2$.
By Thm 3.6(b), $\exists$ a convergent subsequence

TRUE FALSE **b.** For given two real sequences $\{a_n\}$ and $\{b_n\}$, one has

$$\limsup_{n\to\infty} (a_n \cdot b_n) \le \left(\limsup_{n\to\infty} a_n\right) \cdot \left(\limsup_{n\to\infty} b_n\right).$$

Let $a_n = b_n$: $1, -2, 1, -2, 1, -2, 1, -2$ $\quad \limsup = 1$

$a_n \cdot b_n = 1, 4, 1, 4, 1, 4, \ldots$ $\quad \limsup = 4$

TRUE FALSE **c.** Let $f : (X, d_X) \to (Y, d_Y)$ be a continuous function, one-to-one and onto, and X be compact metric space. For every open set U in X, $f(U)$ is open in Y.

Thm 4.17 $\Longrightarrow$ f has an inverse function $f^{-1} : Y \to X$, and f^{-1} is continuous.

read the first paragraph of the proof.

$\forall U \in$ open in X $\quad (f^{-1})^{-1}(U) = f(U)$ is open in Y.

TRUE FALSE **d.** If $g : (X, d_X) \to (Y, d_Y)$ is a continuous function, then $g(\overline{E}) = \overline{g(E)}$ for every set E in X.

$f(x) = e^x : \mathbb{R} \to \mathbb{R}$.

$f(\mathbb{R}) = (0, \infty)$ $\qquad f(\overline{\mathbb{R}}) = f(\mathbb{R}) \neq \overline{(0,\infty)} = [0, \infty)$

TRUE FALSE **e.** $\sum_{n=1}^{\infty} (\sqrt[n]{n} - 1)^n$ is convergent.

Root test: $\sqrt[n]{|a_n|} = \sqrt[n]{|\sqrt[n]{n} - 1|^n} = \sqrt[n]{n} - 1 \to 0$ $\quad$ since $\circledast$ $\sqrt[n]{n} \to 1$

If you do not remember $*$:

$\sqrt[n]{n} - 1 < \frac{1}{2} \Leftarrow \sqrt[n]{n} < \frac{3}{2} \Leftarrow \underbrace{n}_{\text{linear}} < \underbrace{\left(\frac{3}{2}\right)^n}_{\text{exp.}}$ $\forall n \in \mathbb{N}$

$0 < \limsup \sqrt[n]{|a_n|} \le \frac{1}{2} < 1$