1. Provide the definitions of the following concepts for a set E in an arbitrary metric space (X, d). The definitions you give need to be the same (or have the same meaning) as of those definitions given in the textbook or in class, without using a theorem or proposition which requires a proof.

a. A **limit point** of a set E :

A point $p \in X$ is called a limit pt of E if $\forall r>0$ $N_r(p) \cap (E - \{p\}) \neq \phi$.

b. A **bounded set** E :

A set E is called bounded if $\exists R>0 \ \exists p \in X$ s.t. $E \subseteq N_R(p)$.

c. A **boundary point** of a set E :

A point $p \in X$ is called a boundary pt of E if $\forall r>0$ $(N_r(p) \cap E \neq \phi$ and $N_r(p) \cap E^c \neq \phi)$.

d. A **compact set** E :

A set E is called compact if for every open cover $\{G_\alpha \mid \alpha \in \Lambda\}$ of E (that is $E \subseteq \bigcup_{\alpha \in \Lambda} G_\alpha$ and $\forall \alpha$ G_α is open), there exists a finite subcover, that is $\exists \alpha_1, \alpha_2 \cdots \alpha_\ell \in \Lambda$ s.t. $E \subseteq \bigcup_{i=1}^{\ell} G_{\alpha_i}$.

2. Let A and B be countable sets. Prove that $A \times B$ is countable.

You are asked to prove a version of Theorem 2.13. You may use any axiom or proposition/theorem proven earlier, but you can neither use nor refer to Theorem 2.13 and its consequences. Simply, saying "This follows Theorem 2.13" will not earn any credit. You are expected to provide a proof.

For each $a \in A$, let $B_a = \{(a,b) \mid b \in B\}$ (a is fixed).

$\exists$ a bijection $f_a : B_a \to B$, where $f_a((a,b)) = b$,

(f is onto: $\forall b \in B \ \exists (a,b) \in B_a$ s.t. $f_a((a,b)) = b$;
f is 1-1: $f(a,b_1) = f(a,b_2) \Rightarrow b_1 = f(a,b_1) = f(a,b_2) = b_2$.

Since B is countable, B_a is countable:

$$\mathbb{N} \xrightarrow[\text{bij.}]{} B \xrightarrow[f_a^{-1}]{} B_a$$ is a bijection.

$$\bigcup_{a \in A} B_a = \bigcup_{a \in A} \{(a,b) \mid b \in B\} = \{(a,b) \mid a \in A \text{ and } b \in B\} = A \times B.$$

By Thm 2.12, union of countably many countable sets is countable. Hence $\bigcup_{a \in A} B_a = A \times B$ is countable.

3. Prove that every compact set K in a metric space (X, d) is closed.

You are asked to prove Theorem 2.34. You may use any axiom or proposition/theorem proven earlier, but you can neither use nor refer to Theorem 2.34 and its consequences. Simply, saying "This follows Theorem 2.34" will not earn any credit. You are expected to provide a proof.

To prove K is compact, it suffices to prove its complement K^c is open. by Thm 2.23.

Let $p \in K^c$ be chosen arbitrarily.

For any given $q \in K$ define

$$\left.\begin{aligned} V_q &= N_{\frac{1}{2} d(p,q)}(p) \\ W_q &= N_{\frac{1}{2} d(p,q)}(q). \end{aligned}\right\} \begin{array}{c} V_q \cap W_q = \phi \\ \text{by triangle inequality} \end{array}$$

Obs. that $p \notin K$, $q \in K \Rightarrow p \neq q$ and $d(p,q) > 0$.

V_q and W_q are both open by prop 2.19.

$q \in W_q$ and $K \subseteq \bigcup_{q \in K} W_q$, $\mathcal{G} = \{W_q \mid q \in K\}$ is an open cover of K which is compact.

$\exists q_1, q_2, \dots q_\ell$ s.t. $K \subseteq \underbrace{\bigcup_{i=1}^{\ell} W_{q_i}}_{\text{subcover of } \mathcal{G}}$; by defn of compactness

Let $R = \frac{1}{2} \min(d(p,q_1), \dots, d(p,q_\ell)) > 0$, Let $V = N_R(p)$.

$V \subseteq V_{q_i}$ $\forall i = 1, \dots \ell$.

$V \cap W_{q_i} \subseteq V_{q_i} \cap W_{q_i} = \phi$.

$V \cap K \subseteq V \cap \bigcup_{i=1}^{\ell} W_{q_i} = \bigcup_{i=1}^{\ell} (V \cap W_{q_i}) = \phi$.

$V = N_R(p) \subseteq K^c$.

Hence $\forall p \in K^c$ $\exists R > 0$ s.t. $N_R(p) \subseteq K^c$. Hence K^c is open.

K is closed by Thm 2.23.

4. Let E and F be subsets of a metric space (X,d). Let E' denote the set of all limit points of E.

a. Let E' denote the set of all limit points of E. Prove that E' is closed.

Recall Definition: A is closed if $A' \subseteq A$.

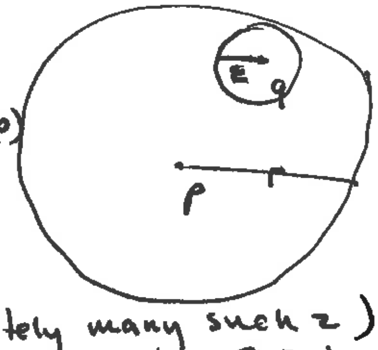
We want to establish $(E')' \subseteq E'$.

Let $p \in (E')'$ be an arbitrary pt. Let $r > 0$ be chosen arbitr.

$N_r(p) \cap (E' - \{p\}) \neq \emptyset$ since $p \in (E')'$.

$\exists q \in N_r(p) \cap (E' - \{p\})$.

$q \in N_r(p)$ which is open. $\exists \varepsilon > 0$ s.t. $N_\varepsilon(q) \subseteq N_r(p)$



$q \in E' \Rightarrow \underbrace{N_\varepsilon(q) \cap (E - \{q\})}_{\text{this is an infinite set by Prop. 2.20}} \neq \emptyset$, and infact,

$\Rightarrow \exists z \in N_\varepsilon(q) \cap E$ s.t. $z \neq q, p$. (actually infinitely many such z exist)

$\exists z \in N_\varepsilon(q) \cap (E - \{p\}) \subseteq N_r(p) \cap (E - \{p\}) \neq \emptyset$.

So $\forall p \in (E')'$ $\forall r > 0$ $N_r(p) \cap (E - \{p\}) \neq \emptyset$. $\forall p \in (E')'$, $p \in E'$.

$(E')' \subseteq E'$

E' is closed

b. Let $\bar{E}$ denote the closure of E. Prove that $\overline{E \cup F} = \bar{E} \cup \bar{F}$.

Recall prop 2.27 (c): (using different letters)

(*) $A \subset B$, B closed $\Rightarrow \bar{A} \subset B$ (proved in class)

Also Recall $\bar{A} = A \cup A'$, so $A \subseteq \bar{A}$.

Proof of (b)

$E \subseteq \bar{E}$

$F \subseteq \bar{F}$

$E \cup F \subseteq \bar{E} \cup \bar{F}$ closed since $\bar{E}, \bar{F}$ closed; prop 2.24

$\overline{E \cup F} \subseteq \bar{E} \cup \bar{F}$ by (*)

$E \subseteq E \cup F \subseteq \overline{E \cup F}$ closed.

$\bar{E} \subseteq \overline{E \cup F}$ by (*)

$\bar{F} \subseteq \overline{E \cup F}$ similarly

$\bar{E} \cup \bar{F} \subseteq \overline{E \cup F}$

$\overline{E \cup F} = \bar{E} \cup \bar{F}$.

See the next page for a proof that many students attempted.

(4b) Another solution

$$\overline{E \cup F} = (E \cup F) \cup (E \cup F)'$$
$$\bar{E} \cup \bar{F} = E \cup E' \cup F \cup F' = E \cup F \cup E' \cup F'.$$

Claim $(E \cup F)' = E' \cup F'$.

(1) Let $p \in E'$ $\forall r > 0$ $N_r(p) \cap (E - \{p\}) \neq \emptyset$

$$\emptyset \neq N_r(p) \cap (E - \{p\}) \subseteq N_r(p) \cap (E \cup F - \{p\}) \neq \emptyset.$$

$\forall r > 0$ $N_r(p) \cap (E \cup F - \{p\}) \neq \emptyset$.

Hence $p \in E' \Rightarrow p \in (E \cup F)'$

$p \in F' \Rightarrow p \in (E \cup F)'$

$E' \cup F' \subseteq (E \cup F)'$.

(2) The reverse inclusion needs more detail.

Let $p \in (E \cup F)'$. Let $n \in \mathbb{N}$ be given.

$$\emptyset \neq N_{\frac{1}{n}}(p) \cap (E \cup F - \{p\}) = \underbrace{\left(N_{\frac{1}{n}}(p) \cap (E - \{p\})\right)}_{A_n} \cup \underbrace{\left(N_{\frac{1}{n}}(p) \cap (F - \{p\})\right)}_{B_n}.$$

Recall example: $\forall n \in \mathbb{N}$ (n is odd, OR n is even)

$\forall n \in \mathbb{N}$ $A_n \cup B_n \neq \emptyset$. $\forall n (A_n \neq \emptyset$ OR $B_n \neq \emptyset)$

Logically: You can't distribute $\forall$ over "OR".

Let $\mathrm{I} = \{n \in \mathbb{N} \mid A_n \neq \emptyset\}$. $\mathrm{I} \cup \mathrm{II} = \mathbb{N}$, an infinite set.

$\mathrm{II} = \{n \in \mathbb{N} \mid B_n \neq \emptyset\}$

Either I or II is an infinite set of natural numbers.

True but not needed: I is infinite $\Rightarrow$ I $= \mathbb{N}$.

Case 1 I is an infinite set $\mathrm{I} = \{\underbrace{n_1, n_2, \dots n_\ell \dots}_{\text{unbounded}}\}$, $\underset{\mathbb{N}}{n_\ell} \to \infty$.

$\forall r > 0$, $\exists n_i \in \mathrm{I}$ s.t. $0 < \frac{1}{n_i} < r$

$N_r(p) \cap (E - \{p\}) \supseteq N_{\frac{1}{n_i}}(p) \cap (E - \{p\}) = A_{n_i} \neq \emptyset$.

$p \in E'$.

Case 2 II is an infinite set ..... $p \in F'$.

$\forall p \in (E \cup F)'$, $p \in E'$ or $p \in F'$. $(E \cup F)' \subseteq E' \cup F'$.

(1) & (2) $\Rightarrow$ $(E \cup F)' = E' \cup F'$

Finally $\overline{E \cup F} = E \cup F \cup (E \cup F)' = E \cup F \cup E' \cup F' = \bar{E} \cup \bar{F}$.

5. TRUE OR FALSE CIRCLE YOUR ANSWERS.
NO PARTIAL CREDITS. YOU ARE NOT EXPECTED TO SHOW WORK.
Correct answers are +4 points each,
wrong answers are -1 point each,
ambiguous answers are -2 points each, and
no answers are 0 point each.
Total of problem 5 will be added to your total grade only if it is positive.
HINT: Read very carefully.

TRUE FALSE **a.** There are non-empty subsets E of $\mathbb{Q}$ such that E is open in $\mathbb{R}$ (with the standard metric $|x-y|$).

If $E \neq \emptyset$, and E is open in $\mathbb{R}$, take $p \in E$.

$\exists r > 0$, $N_r(p) = \underbrace{(p-r, p+r)}_{\text{contains irrational numbers}} \subseteq E$? open.

$E \subseteq \mathbb{Q}$ no rational #s.

TRUE FALSE **b.** The set of real numbers which are not algebraic is an uncountable set.

HW problem #2 p 43. You proved Algebraic numbers, say A are countable. If $\mathbb{R} - A$ were countable, then $\mathbb{R}$ would be countable. Contradiction

TRUE FALSE **c.** For every collection $\{G_\alpha : \alpha \in \Lambda\}$ of open sets in a metric space (X,d), $\bigcap_{\alpha\in\Lambda} G_\alpha$ is open.

$$\bigcap_{n=1}^{\infty} \underbrace{\left(-\frac{1}{n}, \frac{1}{n}\right)}_{\text{open in } \mathbb{R}} = \{0\}$$

not open in $\mathbb{R}$

TRUE FALSE **d.** For a given metric space (X,d), $d_0(x,y) = \frac{d(x,y)}{1+d(x,y)}$ is another metric on X.

Done the same way HW problem #11, p 44, d_5.

See the next page if you didn't do #11: d_5.

TRUE FALSE **e.** If A is a bounded set in an arbitrary metric space (X,d), then $\overline{A}$ is compact since it is ~~closed and~~ bounded.

compact $\Rightarrow$ closed & bounded

"$\Leftarrow$" is false.

5d) $d_0(x,y) = \dfrac{d(x,y)}{1+d(x,y)} \geq 0$ & equality holds $d_0(x,y) = 0$

$\iff d(x,y) = 0$

$\iff x = y.$

$d_0(x,y) = d_0(y,x)$ obvious. since $d(x,y) = d(y,x)$.

Triangle inequality:

$$d_0(x,y) + d_0(y,z) = \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} \geq \frac{d(x,z)}{1+d(x,z)} = d_0(x,z)$$

Set $A = d(x,y)$, $B = d(y,z)$, $C = d(x,z)$ and use the following Lemma.

Lemma: $A, B, C \geq 0$ and $A + B \geq C \implies \dfrac{A}{1+A} + \dfrac{B}{1+B} \geq \dfrac{C}{1+C}$.

Proof

$$\left.\begin{array}{l} AB(C+2) \geq 0 \\ A+B-C \geq 0 \end{array}\right\} \text{add}$$

$$ABC + 2AB + A + B - C \geq 0$$

$$ABC + AB - 1 - C \geq -1 - A - B - AB$$

$$(1+C)(AB-1) \geq -(1+A)(1+B)$$

$$\frac{AB-1}{(1+A)(1+B)} \geq \frac{-1}{1+C}$$

$$\frac{AB-1}{1+A+B+AB} + 1 \geq \frac{-1}{1+C} + 1$$

$$\frac{AB - \cancel{1} + \cancel{1} + A + B + AB}{1+A+B+AB} \geq \frac{C}{1+C}$$

$$\frac{A}{1+A} + \frac{B}{1+B}. = \frac{A(1+B) + B(1+A)}{(1+A)(1+B)} \geq \frac{C}{1+C}$$