1. Provide the definitions of the following concepts in arbitrary metric spaces (X, d). The definitions you give need to be the same (or have the same meaning) as of those definitions given in the textbook or in class, without using a theorem or proposition which require a proof.

a. A **Cauchy sequence**

A sequence $\{p_n\}$ in (X, d) is called Cauchy if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n, m \geq N \quad d(p_n, p_m) < \varepsilon$.

b. A **complete metric space**

A metric space (X, d) is called complete if every Cauchy sequence in (X, d) converges.

c. An **absolutely convergent series** $\sum_{n=1}^{\infty} a_n$

A sequence $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ convergen.

d. A **continuous function** from one metric space into another

A function $f : (X, d_X) \longrightarrow (Y, d_Y)$ is called continuous at a point $p \in X$ if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall x \in X$

$$d_X(p, x) < \delta \Rightarrow d_Y(f(p), f(x)) < \varepsilon.$$

f is called continuous (on X) if it is continuous at each $p \in X$.

e. A **uniformly continuous function** from one metric space into another

A function $f : (X, d_X) \rightarrow (Y, d_Y)$ is called uniformly continuous if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall p, x \in X$

$$d_X(p, x) < \delta \Rightarrow d_Y(f(p), f(x)) < \varepsilon.$$

2. State and prove the **Generalized Mean Value Theorem**.
HINT: It may contain an equation of the form

$$(f(b)-f(a)) \cdot g'(c) = (g(b)-g(a)) \cdot f'(c)$$

a. Statement

Let $f, g : [a,b] \to \mathbb{R}$ be continuous, and let f and g be both diff'ble on (a,b). Then $\exists\, c \in (a,b)$ s.t.

$$(f(b)-f(a)) \cdot g'(c) = (g(b)-g(a))\, f'(c). \quad (*)$$

b. Proof Let $h(t) = (f(b)-f(a)) \cdot g(t) - (g(b)-g(a))\, f(t)$.

$h : [a,b] \to \mathbb{R}$ is continuous by Thm. (4.4)

h is diff'ble on (a,b) by Thm. (5.3)

$h(a) = (f(b)-f(a))\, g(a) - (g(b)-g(a))\, f(a) = f(b)\, g(a) - f(a)\, g(b)$

$h(b) = (f(b)-f(a))\, g(b) - (g(b)-g(a))\, f(b) = -f(a)\, g(b) + g(a) f(b)$

So $h(a) = h(b)$.

By Extreme Value Thm, $h : [a,b] \to \mathbb{R}$ must attain its minimum & maximum values on $[a,b]$. For an interior max or min at c will tell $h'(c) = 0$. However if the max & min are at the boundary, $c = a$ OR $c = b$, we donot have $h'(a)$ & $h'(b)$ defined.

Case 1 h is constant on $[a,b]$. Then $\forall t \in (a,b)$, $h'(t) = 0$.

If h is not constant on $[a,b]$, then either there is an interior max or interior min. (Why? If both max & min values are attained at the boundary, then $h(a) = h(b) = \max = \min$. $\Rightarrow h$ is constant.

Case 2 $\exists\, t \in (a,b)$ s.t. $h(t) > h(a)$. Then max of h is attained at $c \in (a,b)$, $h'(c) = 0$ for some c.

Case 3 $\exists\, t \in (a,b)$ s.t. $h(t) < h(a)$. Then min of h is attained at some $c \in (a,b)$, $h'(c)$.

In all cases $\exists\, c \in (a,b)$ s.t. $h'(c) = 0$.

$h'(c) = (f(b)-f(a))\, g'(c) - (g(b)-g(a))\, f'(c) = 0. \Rightarrow (*)$

3. If both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge absolutely, prove that $\sum_{n=0}^{\infty} c_n$ converges absolutely, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ for all $n \in \mathbb{N} \cup \{0\}$. You must provide a precise statement of every theorem that you use in your solution and we have proven in class or in the textbook.

Thm from the book: (3.50 p. 74). Let $\sum x_n$, $\sum y_n$ be given.

If (i) $\sum_{n=0}^{\infty} x_n$ converges absolutely

(ii) $\sum_{n=0}^{\infty} x_n = A$ & $\sum_{n=0}^{\infty} y_n = B$. ($A$, B both finite)

(iii) $z_n = \sum_{k=0}^{n} x_k y_{n-k}$, then

$$\sum_{n=0}^{\infty} z_n = \left(\sum_{n=0}^{\infty} \sum_{k=0}^{n} x_k y_{n-k} \right) = AB, \text{ i.e. converges}$$

Solution:

$\sum a_n$ abs. conv. $\Rightarrow$ $\sum |a_n|$ converges $= A_0$.

$\sum b_n$ abs. conv $\Rightarrow$ $\sum |b_n|$ converges $= B_0$.

Let $d_n = \sum_{k=0}^{n} |a_k||b_{n-k}|$. By Thm above $\sum_{n=0}^{\infty} d_n$ converges $= A_0 B_0$.

$$0 \le \big| |c_n| \big| = \left| \sum_{k=0}^{n} a_k b_{n-k} \right| \le \sum_{k=0}^{n} |a_k||b_{n-k}| = d_n.$$

By Comparison Test (p 60) $\sum_{n=0}^{\infty} d_n$ converges $\Rightarrow$ $\sum_{n=0}^{\infty} |c_n|$ converges

$\Rightarrow \sum c_n$ converges absolutely.

Caution ① $\left| \sum_{n=0}^{\infty} c_n \right| \le \sum_{n=0}^{\infty} |c_n| \le A_0 B_0 = \sum_{n=0}^{\infty} d_n$

Caution ② $\sum |a_n| = A_0 \Rightarrow \sum a_n$ converges but

$\left| \sum_{n=0}^{\infty} a_n \right| \ne \sum_{n=0}^{\infty} |a_n| = A_0$, unless

all a_n have the same sign.

4. Prove that if $f : (X, d_X) \to (Y, d_Y)$ is a continuous function and X is compact, then $f(X)$ is compact.

You are asked to prove Theorem 4.14 which we proved in class also. You may use any axiom or proposition/theorem proven earlier, but you can neither use nor refer to Theorem 4.14 and the theorem we proved in class or their consequences. Simply, saying "This follows Theorem 4.14 and the theorems we proved in class" will not earn any credit. You are expected to provide proofs.

NOTES: Actually, a stronger result is true. ($E = X$ is your question)

$f : (X, d_X) \to (Y, d_Y)$ continuous, $E \subseteq X$, E compact $\Rightarrow f(E)$ compact.

- Defn (Compact) A set A is called compact, if EVERY open cover of A, has a finite subcover.
- This is NOT COMPACTNESS: A set a has a open cover with finitely many open sets.

Every set $A \subseteq X$ has an open cover by $\{X\}$, since X is open in itself.

PLAN:

In X	② obtain an open cover of E. related to ①	$\longrightarrow$	③ obtain a finite open subcover of E, from ②
	$\uparrow$		$\downarrow$
In Y	① start with an open cover of $f(E)$	$\dashrightarrow$ WTS	④ obtain a finite open subcover of $f(E)$ coming from ①

SOLN / Proof:

Let $\{V_\alpha \mid \alpha \in A\}$ be an open cover of $f(E)$, i.e. V_α open in Y.

$$f(E) \subseteq \bigcup_{\alpha \in A} V_\alpha. \quad \text{Let } U_\alpha = f^{-1}(V_\alpha).$$

$\forall \alpha$ U_α is open in X since V_α is open in Y, & f continuous

$\forall x \in E$, $f(x) \in f(E) \subseteq \bigcup_{\alpha \in A} V_\alpha$. $\exists\, \alpha_0 \in A$ s.t. $f(x) \in V_{\alpha_0}$

$x \in f^{-1}(V_{\alpha_0}) = U_{\alpha_0}$. Hence $E \subseteq \bigcup_{\alpha \in A} U_\alpha$.

PTO

$\{U_\alpha \mid \alpha \in A\}$ is an open cover of E.
E is compact, so there is a finite open subcover:

$$\exists\, \alpha_1, \dots \alpha_\ell \text{ s.t. } E \subseteq \bigcup_{i=1}^{\ell} U_{\alpha_i}$$

$$f(E) \subseteq f\left(\bigcup_{i=1}^{\ell} U_{\alpha_i}\right) = \bigcup_{i=1}^{\ell} f(U_{\alpha_i})$$

$$= \bigcup_{i=1}^{\ell} f(f^{-1}(V_{\alpha_i})) \subseteq \bigcup_{i=1}^{\ell} V_{\alpha_i}$$

Hence the open cover $\{V_\alpha \mid \alpha \in A\}$ for $f(E)$ has a finite open subcover:

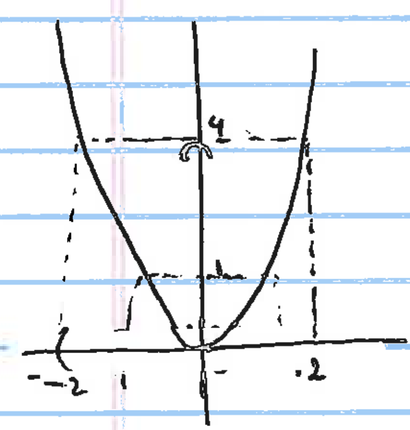
$$f(E) \subseteq \bigcup_{i=1}^{\ell} V_{\alpha_i}.$$

Since this is true for all open covers of $f(E)$, $f(E)$ is compact.

Remark: (U open, f continuous $\not\Rightarrow$ $f(U)$ open.

Ex $f(x) = x^2$ $\quad (-2, 2)$ is open in $\mathbb{R}$
$[0, 4)$ is not open in $\mathbb{R}$.

$$f((-2, 2)) = [0, 4)$$



5. TRUE OR FALSE CIRCLE YOUR ANSWERS.
NO PARTIAL CREDITS. YOU ARE NOT EXPECTED TO SHOW WORK.
Correct answers are +4 points each,
wrong answers are -1 point each,
ambiguous answers are -2 points each, and
no answers are 0 point each.
Total of problem 5 will be added to your total grade only if it is positive.
HINT: Read very carefully.

TRUE FALSE **a.** Given a sequence $\{c_n\}$ in a compact metric space, every subsequence of $\{c_n\}$ converges to the same limit.

$X = [-1,1]$, d = standard Eucl. dist.

$c_n = (-1)^n$ $\exists\, c_{odd} = -1$ $c_{odd} \to -1$

$\exists\, c_{even} = 1$ $c_{even} \to +1$

TRUE FALSE **b.** For given two real sequences $\{a_n\}$ and $\{b_n\}$, one has

$$\liminf_{n\to\infty} (a_n + b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$$

HW question p78 #5

$\limsup (a_n + b_n) \leq \limsup a_n + \limsup b_n$ $\forall$ sequences $\{a_n\}\{b_n\}$

$\limsup (-a_n - b_n) \leq \limsup (-a_n) + \limsup (-b_n)$

$-\liminf (a_n + b_n) \leq -\liminf (a_n) - \liminf (b_n)$

$\liminf (a_n + b_n) \geq \liminf (a_n) + \liminf (b_n)$

TRUE FALSE **c.** If $f : (X, d_X) \to (Y, d_Y)$ is a continuous function, and $\{x_n\}$ is a Cauchy sequence in X, then $\{f(x_n)\}$ is a Cauchy sequence in Y.

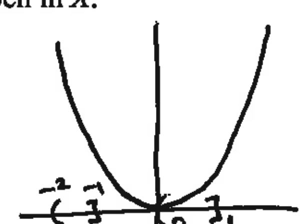
This is true for f uniformly continuous, but NOT for just continuous

Ex $f(x) = \frac{1}{x} : (0,\infty) \to (0,\infty)$ $x_n = \frac{1}{n}$ is Cauchy in $\mathbb{R}$.

$f(x_n) = n$ is NOT Cauchy in $\mathbb{R}$.

TRUE FALSE **d.** If $f : (X, d_X) \to (Y, d_Y)$ is a continuous function, and $f(U)$ is open in Y, then U is open in X.

$f(x) = x^2$:
$f : \mathbb{R} \to \mathbb{R}$



$U = (-2,-1] \cup (0,1]$ not open in $\mathbb{R}$

$f(U) = (0,4)$ open in $\mathbb{R}$.

TRUE FALSE **e.** For every function $g : \mathbb{R} \to \mathbb{R}$ differentiable on all of $\mathbb{R}$, g' satisfies the intermediate value property even if g' is not continuous.

Read Thm 5.12 p 108. It doesnot assume g' is continuous but g' satisfies IV property.

Read line #3 p 109.

p106. Example 5.6(b) is an example for (e).