MATH 4210
Final Exam
May 7, 2018

NAME.______________________________

SIGNATURE.__________________________

Do all **7** problems, **15 points each**.

Show all of your work in order to receive full credit. Every answer must be properly written with logically and grammatically correct sentences and mathematical expressions. **Proofs must have logical continuity and must be mathematically correct**. You need to indicate or state any theorem that you use. The methods of proofs must be indicated, such as induction, proof by contradiction. Show all of your work or indicate its location in the space provided after each problem.

In this test, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ denote the sets of natural numbers, integers, rational numbers and real numbers, respectively.

You are allowed to use any theorem or proposition proved in class or in the textbook, unless the question is asking you to provide a proof of such a theorem. In this case, you may use any axiom, or proposition or theorem proven earlier, but you can neither use nor refer to the theorem you are proving or its consequences. Simply, it does **not** suffice to say ”This is a theorem we have in the book or we have done in class”. You are expected to provide a detailed proof. You can not refer to exercises, examples, or homework, unless you provide a solution to them. When you use a theorem, you can use its name (if name) such as Heine-Borel, or state the fact that you are using. Do not refer to a theorem number, since it is easy to make a mistake with those numbers.

No cell phones (and other communication devices) are allowed to be used during the exam.

DO NOT WRITE BELOW:

1._______ 5. ______

2. ______ 6. ______

3. ______ 7. ______

4. ______

TOTAL.______

1. Provide the definitions of the following concepts in arbitrary metric spaces (X, d). The definitions you give need to be the same (or have the same meaning) as of those definitions given in the textbook or in class, without using a theorem or proposition which require a proof.

a. **Separated sets** and **connected sets**

b. A **convergent sequence**

c. $\overline{\int_a^b} f\,d\alpha, \underline{\int_a^b} f\,d\alpha,$ **and** $\int_a^b f\,d\alpha$. **You may assume the definitions of** $L(P, f, \alpha)$ **and** $U(P, f, \alpha)$.

d. A **sequence of uniformly convergent sequence of functions** $\{f_n\}$

e. **An equicontinuous family of functions**

2. **Prove that closed subsets of compact sets are compact in every metric space** (X, d_X).

You are asked to prove Theorem 2.35 which we proved in class also. You may use any axiom or proposition/theorem proven earlier, but you can neither use nor refer to Theorem 2.35 and the theorem we proved in class or their consequences. Simply, saying "This follows Theorem 2.35 and the theorems we proved in class" will not earn any credit. You are expected to provide proofs.

3. Let $f : (X, d_X) \to (Y, d_Y)$ **be a uniformly continuous function. Prove that** $\{f(x_n)\}$ **is a Cauchy sequence in** Y **for every Cauchy sequence** $\{x_n\}$ **in** X.

4. **Prove that if** f **is a continuous function on** $[a,b]$, **then** $f \in \mathfrak{R}(\alpha)$, **Riemann-Stieltjes integrable with respect to every given monotonically increasing function** α **on** $[a,b]$.

You are asked to prove Theorem 6.8 which we proved in class also. You may use any axiom or proposition/theorem proven earlier, but you can neither use nor refer to Theorem 6.8 and the theorem we proved in class or their consequences. Simply, saying "This follows Theorem 6.8 and the theorems we proved in class" will not earn any credit. You are expected to provide proofs.

5. **For** $n = 1, 2, 3, \ldots, x$ **real**, **put**

$$f_n(x) = \frac{x}{1 + nx^2}.$$

Show that $\{f_n\}$ **converges uniformly to a function** f, **and that the equation**

$$f'(x) = \lim_{n \to \infty} f_n'(x)$$

is correct if $x \neq 0$, **but false if** $x = 0$.

6. Let (X, d_X) be a metric space. Let $\mathfrak{C}(X)$ be the set of all real valued, continuous and bounded functions on X, furnished with the supremum norm $\|f\| = \sup_{x\in X} |f(x)|$, and the metric $d_{\mathfrak{C}}(f,g) = \|f - g\|$. You may assume that $(\mathfrak{C}(X), d_{\mathfrak{C}})$ is a metric space.

Prove that $(\mathfrak{C}(X), d_{\mathfrak{C}})$ **is a complete metric space**.

You are asked to prove Theorem 7.15 which we proved in class also. You may use any axiom or proposition/theorem proven earlier, but you can neither use nor refer to Theorem 7.15 and the theorem we proved in class or their consequences. Simply, saying “This follows Theorem 7.15 and the theorems we proved in class” will not earn any credit. You are expected to provide proofs.

7. **TRUE OR FALSE** CIRCLE YOUR ANSWERS.
NO PARTIAL CREDITS. YOU ARE NOT EXPECTED TO SHOW WORK.
Correct answers are +3 points each,
wrong answers are -1 point each,
ambiguous answers are -2 points each, and
no answers are 0 point each.
Total of problem 5 will be added to your total grade only if it is positive.
HINT: **Read very carefully**.

TRUE FALSE a. **Let** B_n $\forall n \in \mathbb{N}$, **be subsets of a metric space** (X, d_X), **and** $B = \bigcup_{n=1}^{\infty} B_n$. **Then,** $\overline{B} = \bigcup_{n=1}^{\infty} \overline{B}_n$, **where** $\overline{B}$ **is the closure of** B.

TRUE FALSE b. **For a given real sequence** $\{a_n\}$, **it is possible that** $\sum_{n=1}^{\infty} a_n$ **is convergent but** $\sum_{n=1}^{\infty} a_n^2$ **is divergent**.

TRUE FALSE c. **Let** $f = \begin{cases} 0 & if \quad x \in \mathbb{Q} \\ 1 & if \quad x \notin \mathbb{Q} \end{cases} : [0,1] \to [0,1]$. **Then,** $f \in \Re(\alpha)$, **Riemann-Stieltjes integrable with respect to some monotonically increasing function** α **on** $[0,1]$.

TRUE FALSE d. $\sum_{n=1}^{\infty} \frac{\sin nx}{1+e^{nx}}$ converges uniformly on $[1, \infty)$

TRUE FALSE e. **Let** (X, d_X) **be a compact metric space**, **and** $\{f_n\}$ **be a sequence in** $\mathfrak{C}(X)$ **(see Question 6)**. **Then,** $\{f_n\}$ **is uniformly convergent on** X **if and only if** $\{f_n\}$ **is equicontinuous and uniformly bounded**.