SOLUTION Set

MATH 4210
MIDTERM 1
March 6, 2020

NAME.______________________________ SIGNATURE.________________________

Do all **5** problems, **20 points each.**

Show all of your work in order to receive full credit. Every answer must be properly written with logically and grammatically correct sentences and mathematical expressions. **Proofs must have logical continuity and must be mathematically correct.** You need to indicate or state any theorem that you use. The methods of proofs must be indicated, such as induction, proof by contradiction. Show all of your work or indicate its location in the space provided after each problem.

In this test, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ denote the sets of natural numbers, integers, rational numbers and real numbers, respectively.

You are allowed to use any theorem or proposition proved in class or in the textbook, unless the question is asking you to provide a proof of such a theorem. In this case, you may use any axiom, or proposition or theorem proven earlier, but you can neither use nor refer to the theorem you are proving or its consequences. Simply, it does **not** suffice to say "This is a theorem we have in the book or we have done in class". You are expected to provide a detailed proof. You can not refer to exercises, examples, or homework, unless you provide a solution to them. When you use a theorem, you can use its name such as Heine-Borel, or simply state the fact that you are using. **Do not refer to a theorem number**, since it is easy to make a mistake with those numbers.

No cell phones (and other communication devices) are allowed to be used during the exam.

DO NOT WRITE BELOW:

1.______

2.______

3.______

4.______

5.______

TOTAL.______

1. Provide the definitions of the following concepts for a set E in an arbitrary metric space (X, d). The definitions you give need to be the same (or have the same meaning) as of those definitions given in the textbook or in class, without using a theorem or proposition which requires a proof.

a. An **open** set E :

A set E is called open if every point of E is an interior point of E, that is

$\forall p \in E\ \exists r > 0$ s.t. $N_r(p) \subseteq E$.

b. A **closed set** E :

A set E is called closed if every limit point of E is a point of E.

$\forall p \in X \left(\left(\forall r > 0\ N_r(p) \cap (E - \{p\}) \neq \phi \right) \Rightarrow p \in E \right)$

c. A **compact** set E :

A set E is called compact if every open cover of E has a finite subcover of E. For every $\{G_\alpha \mid G_\alpha \text{ open}, \alpha \in \Lambda\}$ s.t. $E \subseteq \bigcup_{\alpha \in \Lambda} G_\alpha$, one can find $\alpha_1, \alpha_2 \ldots \alpha_\ell$ s.t.

$E \subseteq \bigcup_{i=1}^{\ell} G_{\alpha_i}$.

d. A **countable set** E (need not be in a metric space)

A set E is called countable if $E \sim \mathbb{N}$ (or equivalently $\exists$ bijection $f: \mathbb{N} \to E$) where $\mathbb{N}$ is the set of all positive integers

2. Let E be a subset of a metric space (X,d). Prove that E is an open subset of X if and only if its complement E^c is closed in X.

You are asked to prove Theorem 2.23. You may use any axiom or proposition/theorem proven earlier, but you can neither use nor refer to Theorem 2.23 and its consequences. Simply, saying "This follows Theorem 2.23" will not earn any credit. You are expected to provide a proof.

($\Rightarrow$:) Assume E is open.

To prove $(E^c)' \subseteq E^c$, $\forall x \in X \ (x \in (E^c)' \Rightarrow x \in E^c)$

Suffices to prove contrapositive

$\forall x \in X \ (x \notin E^c \Rightarrow x \notin (E^c)')$

Let $x \in X$ s.t. $x \notin E^c$ be given

$x \in E$. E is open. $\exists r > 0 \ N_r(x) \subseteq E$, hence $N_r(x) \cap E^c = \emptyset$

So $\exists r > 0 \quad N_r(x) \cap (E^c - \{x\}) = \emptyset$ which means

not $(\underbrace{\forall r > 0 \ N_r(x) \cap E^c - \{x\} \neq \emptyset}_{x \in (E^c)'})$. So $x \notin (E^c)'$.

($\Leftarrow$:) Assume E^c is closed.

To prove E is open, that is every point p of E is an interior pt.

(*) Suppose Not: $\exists p \in E$, $(\exists r > 0, N_r(p) \subseteq E)$ is false.

$\forall r > 0 \ N_r(p) \nsubseteq E$.

$\forall r > 0 \ N_r(p) \cap E^c \neq \emptyset$.

Since $p \in E$, $E^c - \{p\} = E^c$. $\quad \forall r > 0 \ N_r(p) \cap (E^c - \{p\}) \neq \emptyset$.

$p \in (E^c)'$ by definition

$(E^c)' \subseteq E^c$ since E^c is closed.

$p \in E^c$

This contradicts $p \in E$.

Hence what we supposed in (*) wrong. That is

$\forall p \in E (\exists r > 0, N_r(p) \subseteq E)$.

E is open.

3. Let K be a compact subset of a metric space (X, d). Prove that if E is an infinite subset of K, then E has a limit point in K.

You are asked to prove Theorem 2.37. You may use any axiom or proposition/theorem proven earlier, but you can neither use nor refer to Theorem 2.37 and its consequences. Simply, saying "This follows Theorem 2.37" will not earn any credit. You are expected to provide a proof.

Assume K is compact

$E \subseteq K$, E has infinitely many elements

Want to prove $E' \cap K \neq \phi$.

Suppose $E' \cap K = \phi$.

$\forall q \in K$, one has $q \notin E'$.

not $(\forall r > 0 \quad N_r(q) \cap (E - \{q\}) \neq \phi)$

$\exists r > 0 \quad N_r(q) \cap (E - \{q\}) = \phi$.

$N_r(q) \cap E \subseteq \{q\}$. $\quad$ r depends on q. So we use r_q

$$K \subseteq \bigcup_{q \in K} N_{r_q}(q)$$

K compact $\Rightarrow \exists q_1, q_2, \dots, q_\ell$ s.t. $K \subseteq \bigcup_{i=1}^{\ell} N_{r_{q_i}}(q_i)$

$$\underset{E \subseteq K}{E = E \cap K} \subseteq E \cap \left(\bigcup_{i=1}^{\ell} N_{r_{q_i}}(q_i) \right) = \bigcup_{i=1}^{\ell} (E \cap N_{r_{q_i}}(q_i))$$

$$\subseteq \bigcup_{i=1}^{\ell} \{q_i\}$$

$$= \{q_1, q_2, \dots q_\ell\}.$$

$\underset{\text{infinite}}{E} \subseteq \underbrace{\{q_1, q_2, \dots q_\ell\}}_{\text{finite}}$. This cannot happen. Contradiction.

Conclude $E' \cap K \neq \phi$.

4. Consider the set of rational numbers $\mathbb{Q}$ as a metric space with $d(p,q) = |p-q|$. Let $E = \{p \in \mathbb{Q} : p > 0 \text{ and } 2 < p^2 < 5\}$.

You may assume without proof that for all real numbers a, b with $a < b$, one has that $\mathbb{Q} \cap (a,b)$ is open in $\mathbb{Q}$, and $\mathbb{Q} \cap [a,b]$ is closed in $\mathbb{Q}$.

a) Prove that E is not compact.

b) Prove that E is not connected.

(a) Let $G_n = (\sqrt{2}+\frac{1}{n}, \sqrt{5}) \cap \mathbb{Q}$ for $n \geq 2$, $n \in \mathbb{N}$.

G_n is open in $\mathbb{Q}$, given.

Given any $x \in (\sqrt{2},\sqrt{5}) \cap \mathbb{Q}$, $\sqrt{2} < x < \sqrt{5}$.

$\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < x - \sqrt{2}$ (Arch. Prin.)

$\sqrt{2} + \frac{1}{n} < x < \sqrt{5}$, so $x \in G_n$

$\forall x \in E$ $x \in \bigcup_{n=1}^{\infty} G_n$. So $E \subseteq \bigcup_{n=1}^{\infty} G_n$ (Actually $=$)

$\{G_n | n \in \mathbb{N}\}$ is an open cover of E.

Let $\{G_{n_1}, G_{n_2}, \ldots G_{n_\ell}\}$ be a subcollection of $\{G_n | n \in \mathbb{N}\}$.

Let $L = \max(n_1, n_2, \ldots n_\ell)$.

$\bigcup_{i=1}^{\ell} G_{n_i} = G_L$ since $\forall n$ $G_{n+1} \supseteq G_n$.

$\exists r \in \mathbb{Q}$ s.t. $\sqrt{2} < r < \sqrt{2} + \frac{1}{L}$ by density of rationals.

$r \in E$, but $r \notin G_L$. $E \not\subseteq G_L$. Hence $\{G_n | n \in \mathbb{N}\}$ is a cover of E, but it has no finite subcover for E.

E is not compact.

(b) Let $F = \{p \in \mathbb{Q} \mid p > 0, 2 < p^2 < 3\} = (\sqrt{2},\sqrt{3}) \cap \mathbb{Q} \neq \emptyset$ since $1.5 \in F$

$G = \{p \in \mathbb{Q} \mid p > 0, 3 < p^2 < 5\} = (\sqrt{3},\sqrt{5}) \cap \mathbb{Q} \neq \emptyset$ since $2 \in G$.

$F \cup G = ((\sqrt{2},\sqrt{3}) \cup (\sqrt{3},\sqrt{5})) \cap \underset{\sqrt{3}}{\underset{\not\ni}{\mathbb{Q}}} = (\sqrt{2},\sqrt{5}) \cap \mathbb{Q} = E$.

$F = \underbrace{(\sqrt{2},\sqrt{3}) \cap \mathbb{Q}}_{\text{open in } \mathbb{Q}} = \underbrace{[\sqrt{2},\sqrt{3}] \cap \mathbb{Q}}_{\text{closed in } \mathbb{Q}} = \overline{F}$ since $\overline{F}$ is the smallest closed set containing F, but F is closed. So $\overline{F} = F$. $G = \overline{G}$

$F \cap G = ((\sqrt{2},\sqrt{3}) \cap (\sqrt{3},\sqrt{5})) \cap \mathbb{Q} = \emptyset$.

$\overline{F} \cap \overline{G} = \overline{F} \cap G = F \cap \overline{G} = F \cap G = \emptyset$.

We have a non-trivial seperation of E

E is not connected

5. **TRUE OR FALSE** CIRCLE YOUR ANSWERS.

NO PARTIAL CREDITS. YOU ARE NOT EXPECTED TO SHOW WORK.

Correct answers are +4 points each,
wrong answers are -1 point each,
ambiguous answers are -2 points each, and
no answers are 0 point each.

Total of problem 5 will be included if your total grade only if it is positive. If total of Problem 5 is negative, it will be treated as zero.

HINT: Read very carefully.

TRUE FALSE **a.** The set of irrational numbers is an uncountable set.

$\mathbb{R}$ uncountable, $\mathbb{Q}$ is countable.

If $\mathbb{R}-\mathbb{Q}$ were countable, then $\mathbb{R}=\mathbb{Q}\cup(\mathbb{R}-\mathbb{Q})$ would be countable, not the case.

$\mathbb{R}-\mathbb{Q}$ is uncountable.

TRUE FALSE **b.** For every subset E of a metric space (X,d), one has $(\overline{E})^\circ = E^\circ$, where E° is the set of all interior points of E.

$E=(0,1)\cup(1,2)\subseteq\mathbb{R}$. standard metric

$\overline{E}=[0,2]$.

$(\overline{E})^\circ=(0,2)\neq E^\circ=E=(0,1)\cup(1,2)$

TRUE FALSE **c.** For all subsets A and B of a metric space (X,d), one has $\overline{A}\cap\overline{B}=\overline{A\cap B}$ and $\overline{A}\cup\overline{B}=\overline{A\cup B}$.

$A=(0,1)$, $B=(1,2)$ $\subseteq\mathbb{R}$ standard metric

$\overline{A}=[0,1]$, $\overline{B}=[1,2]$

$\overline{A}\cap\overline{B}=\{1\}\neq\overline{A\cap B}=\overline{\emptyset}=\emptyset$.

TRUE FALSE **d.** Let X be an infinite set. For all p and q in X, define

$$d_0(p,q)=\begin{cases}1 & \text{if } p\neq q\\ 0 & \text{if } p=q\end{cases}$$

. Then, every subset of (X,d_0) is closed and open, and consequently, each connected subset has at most one point. (i) $\{x\}$ is open: $\{x\}=N_{\frac{1}{2}}(x)$

Every set $A=\bigcup_{x\in A}\{x\}$ is open, since union of open sets.

Every set A is closed since its complement is open.

If $A\cap B=\emptyset$, then $\overline{A}\cap B=A\cap\overline{B}=\overline{A}\cap\overline{B}=\emptyset$. If E is connected, $p\neq q\in E$: take $A=\{p\}$ take $B=E-\{p\}$ for seperation

TRUE FALSE **e.** If A is a bounded set in $\mathbb{R}^k$ with the standard metric $d(p,q)=|p-q|$, then $\overline{A}$ is compact.

A bdd $\Rightarrow\exists R$ $A\subseteq B_R(0)$.

$\Rightarrow\overline{A}\subseteq\overline{B_R(0)}=\{x\in\mathbb{R}^k \mid \|x\|\leq R\}\subseteq B_{R+1}(0)$ So $\overline{A}$ is bounded.

$\overline{A}$ closed. Heine Borel $\Rightarrow\overline{A}$ compact.