Review Session 11/5/2019

3.3 #10

① • Density of Rationals: $\forall x,y \in \mathbb{R}$

$x<y \Rightarrow \exists r \in \mathbb{Q}$ s.t. $x<r<y$.

Proved in class.

② • Density of Irrationals $\forall x,y \in \mathbb{R}$

$x<y \Rightarrow \exists a \in \mathbb{R}-\mathbb{Q}$ s.t. $x<a<y$.

① $\Rightarrow$ ② $x<y$, $x,y \in \mathbb{R}$ be given

$x\sqrt{2} < y\sqrt{2}$ $x\sqrt{2}, y\sqrt{2} \in \mathbb{R}$.

by ① $\exists r \in \mathbb{Q}$ s.t. $x\sqrt{2} < r < y\sqrt{2}$

$x < \frac{r}{\sqrt{2}} < y$

if $r \neq 0$, then $\frac{r}{\sqrt{2}} \notin \mathbb{Q}$. Take $a = \frac{r}{\sqrt{2}}$

if $r=0$, then $x\sqrt{2} < 0 < y\sqrt{2}$

$\exists r' \in \mathbb{Q}$, $0 < r' < y\sqrt{2}$

$0 < \frac{r'}{\sqrt{2}} < y$

then take $a = r'/\sqrt{2} \notin \mathbb{Q}$.

To do Exc 3.3 # 10a

Proof by induction

Let $P(n)$: $\forall x,y$ ($x<y$) $\exists r_1, r_2, r_3, \dots r_n \in \mathbb{Q}$ s.t

$x < r_1 < r_2 < r_3 < r_4 < \dots < r_n < y$

$P(1)$ $\forall x<y$ $\exists r_1 \in \mathbb{Q}$ $x<r_1<y$ by density of rationals.

WTS $P(n) \Longrightarrow P(n+1)$

Assume $P(n)$ $\forall x<y$ $\exists r_1 \dots r_n$ s.t. $x<r_1<r_2<\dots<r_n<y$

Density of rationals $\exists r_{n+1} \in \mathbb{Q}$ $r_n<r_{n+1}<y$

$\forall x<y$ So $\exists r_1, r_2, \dots r_{n+1} \in \mathbb{Q}$ s.t.

$x<r_1<r_2<\dots<r_n<r_{n+1}<y$: $P(n+1)$.

All of r_i are distinct for $i=1,2,3,\dots$ $i \in \mathbb{N}$.

we found infinitely many rationals between x & y

10b Almost identical proof

Density of irrational (*) Assume $\forall x,y \in \mathbb{R}$ ___ $x<y \Rightarrow \exists a_1 \in \mathbb{R}-\mathbb{Q}$ s.t. $x<a_1<y$

Let $Q(n)$: $\forall x,y \in \mathbb{R}$ $\exists$ $a_1, a_2 \dots a_n$ all irrational s.t. $x<a_1<a_2<\dots<a_n<y$

Proof by Induction.

$Q(1)$ proved (Density of irrationals)

WTS $Q(n) \Rightarrow Q(n+1)$

$\forall x<y$ $\exists \underbrace{a_1 \dots a_n}_{\text{all irrational}}$ $x<a_1<\dots<a_n<y$. $Q(n)$

$\exists$ a_{n+1} irrational $a_n<a_{n+1}<y$

$x<a_1<\dots<a_n<a_{n+1}<y$ $Q(n+1)$

$\forall i \in \mathbb{N}$ All a_i are distinct, all $a_i \in \mathbb{R}-\mathbb{Q}$. #

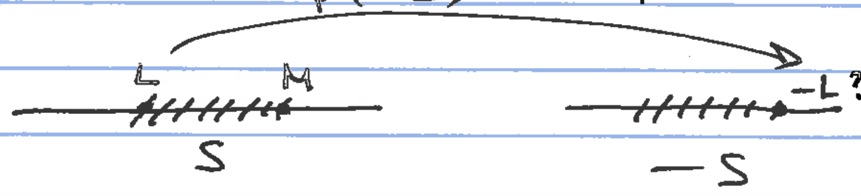
3.3 7b.

a) $k \geq 0$ then $\sup kS = k \sup S$

7b Do this $k = -1$ first.

i.e. Want $\sup(-S) = -\inf S$

Idea



S bounded, $S \neq \emptyset$ $\quad \underset{M}{\underset{\parallel}{\sup S}}, \underset{L}{\underset{\parallel}{\inf S}}$ both exist.

Want $\sup(-S) = -L$

Defn of sup.

(i) Need to show $-L$ is an upper bd for $-S$

(ii) " " " any $N < -L$, N is not an upper bound of $-S$.

(i) $\forall\, x \in S \quad x \geq L = \inf S$

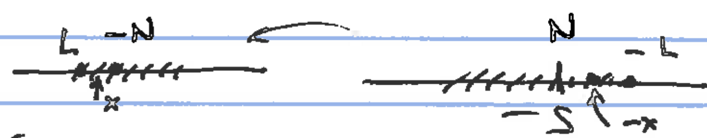
$\forall\, y \in -S$, $y = -x$ for some $x \in S$

$y = -x \leq -L$

$-L$ is an upper bound for $-S$.

$\Leftarrow$ $-L = \sup(-S)$

(ii)



$\forall\, N < -L$

$-N > L = \inf S$

$-N$ is not a lower bd for S

$\exists\, x \in S \quad L \leq x < -N$

$-x > N, \quad -x \in -S$

N can't be an upper bound for $-S$.

7b)

Am I allowed to use 7(a)?

YES: Assume $\begin{cases} \inf(kS) = k \inf S & k>0 \quad ① \\ \sup(kS) = k \sup S & k>0 \end{cases}$

proved $\sup(-S) = -\inf S$. ③

For $m = -k < 0$, $k>0$

$\sup(mS) = \sup(-kS) \underset{\text{by } ③}{=} -\inf(kS) \underset{①}{=} -k \inf S$

$= m \inf S$

What if I am not allowed to use (a).

PTO

7b Case $k < 0$ without assuming 7a.

WTS $\sup(kS) = k \inf S$

S bounded, $S \neq \emptyset$ $\quad$ $\sup S$, $\inf S$ both exist.

$$\underset{M}{\sup S} = , \quad \underset{L}{\inf S} =$$

Want : $\sup(kS) = kL$

From def'n of supremum.
(i) Need to show kL is an upper bd for kS
(ii) Need to show any $N < kL$, N is not an upper bd of kS.

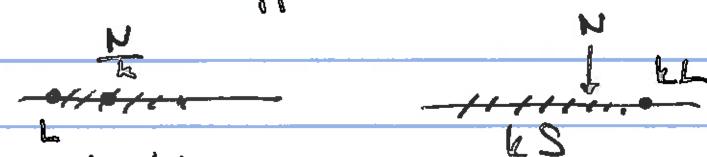
(i) $\forall x \in S$ $\quad x \geq L = \inf S$

$\forall y \in kS$ $\quad y = kx$ for some $x \in S$

$y = kx \leq kL$ $\quad (k<0)$

kL is an upper bd for kS.

(ii)



for any $N < kL$:

$N < kL \implies \frac{N}{k} > L$ $\quad (k<0)$

Def'n of infimum: $\frac{N}{k}$ is not a lower bd for S

$\exists x \in S$ s.t. $L \leq x < \frac{N}{k}$.

$kx > N$, $kx \in kS$

N can't be an upper bd for kS.

$kL = \sup(kS)$

$L = \inf S$

$\sup(kS) = k \inf S$ #.

Old Exam 2002

3) $S = [1,2)$

$\text{int } S = (1,2)$

$[1,2] = S'$, $2 \in S'$.

$[1,2] = \bar{S} = cl(S)$

S is not open

S is not closed

$bd\, S = \{1,2\}$

In our test, you do not need to prove interior, accumulation pt, open, closed. (section 3.4 only).

HOWEVER There will be proofs from all other sections.

Exam #5 (e)

CORRECTION:

As it appears in the handout

If a sequence (s_n) in $\mathbb{R}$ satisfies
$\forall m, n \in \mathbb{N}$, $\underline{\underline{n < m}}$, $|s_n - s_m| \le \frac{1}{n}$:

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$, $N > \frac{1}{\varepsilon}$
$\forall n, m \ge N$, $m > n \ge N$
$|s_n - s_m| \le \frac{1}{n} \le \frac{1}{N} < \varepsilon$
$\Big\}$ (s_n) is Cauchy

Hence (s_n) converges in $\mathbb{R}$, since in $\mathbb{R}$ every Cauchy sequence converges. TRUE

In the test there was a correction of typo & question asked (17 yrs ago) was

$\forall m, n \in \mathbb{N}$ $\underline{\underline{m < n}}$ $|s_n - s_m| \le \frac{1}{n}$.

Then "s_n converges" is still TRUE

Why?

Take $m = 1$ $\forall n$ $1 < n$, $|s_n - s_1| \le \frac{1}{n}$

$$s_1 - \frac{1}{n} \le s_n \le s_1 + \frac{1}{n}$$

$\lim s_n = s_1$ $\quad$ $s_1 - \frac{1}{n} \to s_1$, $s_n \to s_1$, $s_1 + \frac{1}{n} \to s_1$ $\quad$ let $n \to \infty$

Take $m = 2$. $\forall n$ $n > 2 = m$ $|s_n - s_2| \le \frac{1}{n}$

So $\lim s_n = s_2$

similarly $\lim s_n = s_3$ & so on.

$s_1 = s_2$
$s_2 = s_3$
$s_3 = s_4$
$\vdots$

$\Longrightarrow$ (s_n) is a constant sequence.