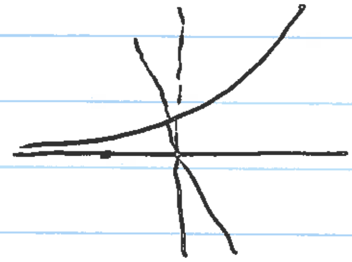
2006 exam 5b/c

$$e^x = -3x$$



$$f(x) = 3x + e^x$$

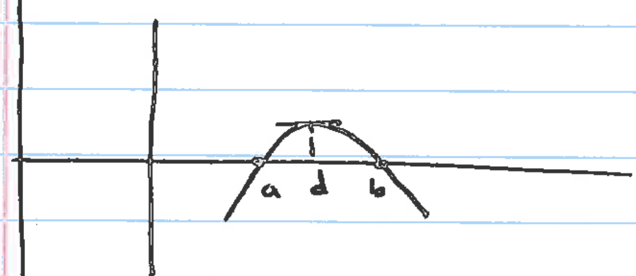
$$f(0) = 1$$

$$f(-1) = -3 + e^{-1} = -3 + \underbrace{\frac{1}{e}}_{<1} < -2$$

IVTh $\exists\, c$ s.t. $f(c) = 0$

$$3c + e^c = 0$$

$$e^c = -3c$$



Recall g diff'ble

$g(a) = g(b) = 0$

Rolle's Thm

$\exists\, d \in (a,b)$ $g'(d) = 0$

Suppose $\exists\, c_1, c_2$ s.t. $f(c_1) = f(c_2) = 0$

$\Longrightarrow \exists\, d$ s.t. $f'(d) = 0$

$d \in (c_1, c_2)$

$$f(x) = 3x + e^x$$

$$f'(x) = 3 + \underbrace{e^x}_{>0} > 3 > 0$$

Hence $\exists$ no such d: $f'(d) = 0$. Hence $\exists$ only one c $\overbrace{\ }^{f(c)=0}$

2006(# 3) $h(x) = \begin{cases} x^2 \cos\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

$$\lim_{x\to 0} \frac{h(x)-h(0)}{x-0} = \lim_{\substack{x\to 0 \\ x \neq 0}} \frac{x^2\cos\frac{1}{x} - 0}{x-0} = \lim_{x\to 0} x\cos\frac{1}{x} = \underset{h'(0)}{\overset{\parallel}{0}}$$

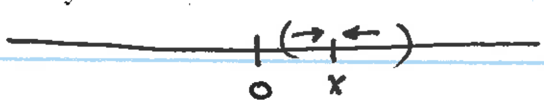
because

$\forall \varepsilon > 0 \ \exists \delta = \varepsilon \quad \forall x \in \mathbb{R}, \ |x-0| < \delta \Rightarrow$

$$\left|x\cos\frac{1}{x}\right| \leq |x| < \varepsilon$$

$$h'(x) = \begin{cases} 2x\cos\frac{1}{x} - \left(\sin\frac{1}{x}\right)\left(\frac{-1}{x^2}\right)\cdot x^2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

when $x \neq 0$ $\exists \ \varepsilon > 0$ s.t. $N_\varepsilon(x) \subseteq \underbrace{\mathbb{R} - \{0\}}_{\text{open}}$



so we can calculate $h'(x)$ for $x \neq 0$ by using $x^2\cos\frac{1}{x}$ only

$$\lim_{x\to 0} h(x) = 0 = h(0)$$

$$\boxed{\underbrace{\lim_{x\to 0} h'(x)}_{\text{DNE}}} \neq h'(0) = 0$$

exam 2006

2b $\left.\begin{array}{l} f: I \to J \\ g: J \to \mathbb{R} \\ I, J \text{ intervals} \\ f \text{ cont at } c \\ g \text{ cont at } f(c) \end{array}\right\} \implies g \circ f$ cont at c.

$$\boxed{\begin{array}{l} \cdot \text{ Recall Thm: } f: D \to \mathbb{R},\ c \in D \\ f \text{ cont at } c \iff \forall (s_n),\ s_n \in D,\ s_n \to c, \text{ one has } \lim_{n\to\infty} f(s_n) = f(c) \end{array}}$$ *

Let (s_n) be any sequence in I s.t. $s_n \to c$

f cont at c given

$(* \Rightarrow)$ $\lim_{n\to\infty} f(s_n) = f(c)$

Let $d_n = f(s_n)$, and then $d_n \to f(c)$

g continuous at $f(c)$

$(* \Rightarrow)$ $\lim g(d_n) = g(f(c))$

$\lim_{n\to\infty} g(f(s_n)) = \lim_{n\to\infty} (g \circ f)(s_n) = (g \circ f)(c)$

Since this is true for all $(s_n) \in I$, $s_n \to c$

$(\overset{*}{\Leftarrow})$ $g \circ f$ is continuous at c.

$$\begin{array}{ccccc} I & \xrightarrow{f} & J & \xrightarrow{g} & \mathbb{R} \\ c & & f(c) & & \end{array}$$

Same question via $\varepsilon - \delta$ definition

for g at $f(c)$: $\forall \varepsilon > 0 \ \exists \eta > 0 \ \forall y \in J \quad |y - f(c)| < \eta \Rightarrow |g(y) - g(f(c))| < \varepsilon$

for f at c: given $\eta \quad \exists \delta > 0 \ \forall x \in I \quad |x - c| < \delta \Rightarrow |f(x) - f(c)| < \eta$

Combining:

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in I \quad |x - c| < \delta \Longrightarrow |\underbrace{f(x)}_{= y} - f(c)| < \eta$$

$$|y - f(c)| = |f(x) - f(c)| < \eta$$

$$\Longrightarrow |g(f(x)) - g(f(c))| < \varepsilon$$

$$\Longrightarrow |(g \circ f)(x) - (g \circ f)(c)| < \varepsilon$$

Continuity of $g \circ f$ at c.

HW #12 §5.2 $f: D \to \mathbb{R}$ $f(x) \geq 0$

$\sqrt{f}: D \to \mathbb{R}$ s.t. $\sqrt{f}(x) = \sqrt{f(x)}$.

remember

$$|\sqrt{a} - \sqrt{b}| = \left|\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}} \cdot \sqrt{a}-\sqrt{b}\right| = \frac{|a-b|}{\sqrt{a}+\sqrt{b}}. \leq \frac{|a-b|}{\sqrt{a}}$$

b changing
a fixed if $a > 0$.

If $a = 0$

$$|\sqrt{a} - \sqrt{b}| = |\sqrt{b}| < \underbrace{\sqrt{\delta}}_{= \varepsilon} \Leftarrow |b| < \delta$$

Use sequential characterization

STEP 1.

$(s_n) \subseteq (0,\infty)$, $\forall s_n \geq 0$, $s_n \to c \geq 0 \implies \sqrt{s_n} \to \sqrt{c}$.

Case 1. $c > 0$. Let $\varepsilon > 0$ $\exists N$ then $(n > N \Rightarrow |s_n - c| < \varepsilon\sqrt{c}$

$$\forall n \geq N \quad |\sqrt{s_n} - \sqrt{c}| = \left|\frac{\sqrt{s_n}+\sqrt{c}}{\sqrt{s_n}+\sqrt{c}} \cdot \sqrt{s_n}-\sqrt{c}\right| = \frac{|s_n - c|}{\sqrt{s_n}+\sqrt{c}} \leq \frac{|s_n - c|}{\sqrt{c}} < \varepsilon$$

Hence $\sqrt{s_n} \to \sqrt{c}$.

Case 2 $c = 0$ Let $\varepsilon > 0$ be given

$\exists N$ $\forall n \geq N$ $(n > N \implies |s_n - 0| < \varepsilon^2)$

$$\forall n \geq N \quad |\sqrt{s_n} - 0| = |\sqrt{s_n}| = \sqrt{s_n} < \varepsilon. \qquad \sqrt{s_n} \to \sqrt{c}.$$

Recall Thm $f: D \to \mathbb{R}$

f cont at c $\iff$ $\forall (s_n)$, $s_n \in D$, $s_n \to c$,
one has
$$\lim_{n\to\infty} f(s_n) = f(c)$$

Take $f = \sqrt{x}$

$\sqrt{x}$ cont at c $\iff$ $\forall (s_n)$, $s_n \in [0, \infty)$ $s_n \to c$
one has
$$\lim_{n\to\infty} \sqrt{s_n} = \sqrt{c}$$

This is done in STEP 1.

STEP 2 Since we know $g(x) = \sqrt{x}$ is continuous; Now we can prove HW #12 5.2 p 214

$$\begin{tikzcd} c \in D \arrow[r, "f"] \arrow[rr, bend right, "g \circ f = \sqrt{f(x)}"'] & \mathbb{R} \arrow[r, "\sqrt{x} = g"] & \mathbb{R} \end{tikzcd}$$

The continuity of $\sqrt{f(x)}$ at c follows Thm 5.2.12
see pages 3 or 4 of this review.

2004 $7\overset{a}{b}$

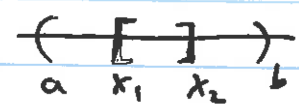
MVT: Let $f : [a,b] \to \mathbb{R}$, continuous
f diff'ble on (a,b)
$\Rightarrow \exists\ c \in (a,b)$ s.t. $f'(c) = \dfrac{f(b) - f(a)}{b-a}$.

Question
$f : (a,b) \longrightarrow \mathbb{R}$ diff'ble on (a,b)

$\forall x\ f'(x) \geq 0 \iff f \uparrow$ on (a,b).

$\Rightarrow$: use MVT
$\Leftarrow$: use limit theorems.

$(\Rightarrow :)$ Let $x_1, x_2 \in (a,b)$ s.t. $a < x_1 < x_2 < b$
use MVT on $[x_1, x_2]$.



$\exists\ c$ (depending on x_1, x_2, f)

s.t. $f'(c) = \dfrac{f(x_1) - f(x_2)}{x_1 - x_2}$.

Hypothesis $0 \leq f'(c) \Rightarrow \dfrac{f(x_1) - f(x_2)}{x_1 - x_2} \geq 0$

$x_1 - x_2 < 0$

$f(x_1) - f(x_2) \leq 0$

$\left(\begin{array}{c}\text{started with} \\ x_1 < x_2\end{array}\right)$ $\quad f(x_1) \leq f(x_2) \quad f \uparrow$.

$(\Leftarrow:)$ $f \uparrow$ on (a,b) given

Let $c \in (a,b)$

$f'(c) = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$. exists since f is diffble at c

Case 1 $\quad x > c \quad f(x) \geq f(c) \quad \dfrac{f(x) - f(c)}{x - c} \geq 0 \quad \dfrac{+}{+}$

Case $\quad x < c \quad f(x) \leq f(c) \quad \dfrac{f(x) - f(c)}{x - c} \geq 0 \quad \dfrac{-}{-}$

$\forall x \neq c \quad \dfrac{f(x) - f(c)}{x - c} \geq 0$

$\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} \geq 0$ via Thms 4.2.4, 4.2.5 5.1.8

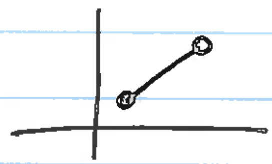
$f'(c) \geq 0$.

T/F questions

2006 (a) T

(b) T closed + bounded $\Rightarrow$ compact in $\mathbb{R}$

(c) F



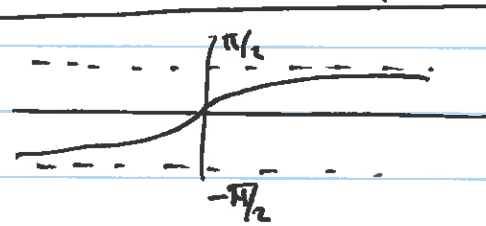
(d) T (Thm)

(e) T bounded & contains its accumulation pts.

(f) T Cauchy in $\mathbb{Q}$ $\Rightarrow$ Cauchy in $\mathbb{R}$

$\Rightarrow$ convergent in $\mathbb{R}$

2005 a) F



$\tan^{-1}x$

b) T Finite sets have no accumulation pts.

c) T $B = \{x_1, \ldots x_n\}$

$f(B) = \{f(x_1), \ldots f(x_n)\}$ is a finite set

with a largest & smallest elt.

d) F 001001001001001 --

110110110110110 ---

e) T (Thm) ($\Leftrightarrow$ Int Val. Thm)

f) T

2004

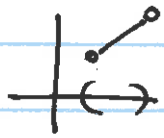
a) F $\quad 0 \notin$ Set $\quad 0$ is an accum. point

b) F $\quad$ it is possible to have

$$x_n, y_n \to +\infty$$

but $f(x_n) \to +\infty$, $f(y_n) \to -\infty$ still have $\frac{1}{f} \to 0$

c) F



d) T

e) T $\quad$ since $\mathbb{N}' = \phi$.

2003

a) F $\quad 0 \notin$ set, 0 accum pt., Not closed

b) T

c) T $\quad \mathbb{N}$

d) T $\quad 0, 8, 0, 32, 0, 72, 0,$

$\exists (0)$ subsequence

e)

f) F $\quad \phi$ is open & compact

2002

a) T

b) F $\quad$ limit can be irrational.

c) T $\quad \mathbb{N}$

d) F $\quad [0,1)$

e)

f)