Dec 11
Review
1

2006 (1a)

$$\lim_{x\to 0} \left|\sin \frac{1}{x}\right|$$

prep. $\begin{cases} x_n = \dfrac{1}{2\pi n} & \dfrac{1}{x_n} = 2\pi n. \\ & \left|\sin \dfrac{1}{x_n}\right| = 0 \quad \text{if } n \text{ is even} \\ x_n = \dfrac{1}{2\pi n + \frac{\pi}{2}} & \left|\sin \dfrac{1}{x_n}\right| = 1 \quad \text{if } n \text{ is odd} \end{cases}$

Soln

Let $x_n = \begin{cases} \dfrac{1}{2\pi n} & \text{if } n \text{ is even} \\ \dfrac{1}{2\pi n + \frac{\pi}{2}} & \text{if } n \text{ is odd} \end{cases} \longrightarrow 0$

$f(x) = \left|\sin \frac{1}{x}\right|$

$f(x_n)$: $1, 0, 1, 0, 1, \ldots$

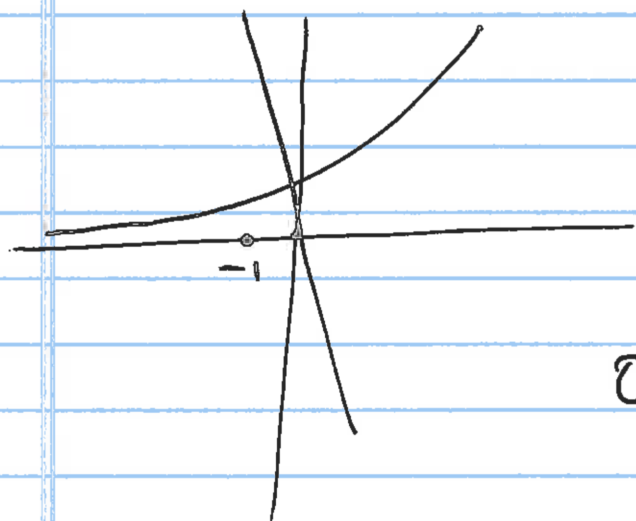
$\lim f(x_n)$ DNE.

By Thm 5.1.10, $\lim_{x\to 0} \left|\sin \frac{1}{x}\right|$ DNE

In the exam, you need to write the statement of each theorem you use.

(2006) #5 b, c

$e^x = -3x$ has a soln. on $\mathbb{R}$



$f(x) = e^x + 3x$

$f(-1) = e^{-1} - 3 = \frac{1}{e} - 3 < 0.$

$f(0) = e^0 + 0 = 1. > 0$

On the interval $[-1, 0]$

$f(-1) < 0$

$f(0) > 0$

$k = 0$ is between $f(-1)$ & $f(0)$.

By IVT $\exists c \in (-1, 0)$ s.t. $f(c) = 0$

$f(c) = e^c + 3c = 0$

$e^c = -3c$

Suppose $\exists c_1 \neq c_2$ s.t. $f(c_1) = f(c_2) = 0$

By MVT $\exists c$ between c_1 & c_2 s.t.

$$f'(c) = \frac{f(c_1) - f(c_2)}{c_1 - c_2} = 0$$

But, $f' = (e^x + 3x)' = e^x + 3 \geq 3 \ \forall x \in \mathbb{R}$

$f' \neq 0$ for any c.

So one can't have two distinct solutions $c_1 < c_2$ s.t. $f(c_1) = f(c_2) = 0$.

2006 (3a) $f(x) = \begin{cases} x^2 \cos\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

Does it exist? $\displaystyle\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^2\cos\frac{1}{x} - 0}{x-0} = \lim_{x\to 0} x\cos\frac{1}{x}$

$= 0,$

since

$\forall \varepsilon > 0 \ \exists \delta > 0 \ (\delta = \varepsilon)$ s.t.

$\forall x \in \mathbb{R}$

$|x-0| < \delta \Rightarrow \left|x\cos\frac{1}{x} - 0\right| \leq |x| \underbrace{\left|\cos\frac{1}{x}\right|}_{\leq 1} \leq |x| < \delta = \varepsilon.$

So $f'(0) = 0.$

2006 T/F

(a) T

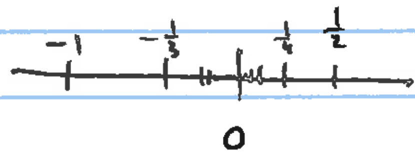
(b) T

(c) F $\quad f(x)=\frac{1}{x} : (1,2) \longrightarrow (\frac{1}{2}, 1)$

$\frac{1}{2} < f < 1$

(d) T Thm

(e) T



Compact / not open

bdd closed $S'=\{0\}$

(f) T Every Cauchy sequence in $\mathbb{Q}$ is a Cauchy sequence in $\mathbb{R}$.

Thm: Hence it converges to a limit in $\mathbb{R}$.

2005 T/F

a) F $\quad f(x)\equiv 1 : \mathbb{R} \to \{1\}$

b) T $\quad$ (if S is finite, then $S'=\emptyset$)

c) T $\quad$ Choose largest of $f(x_1), f(x_2), \dots, f(x_n)$

smallest " " " ".

False d) $\{0,1,0,1,0,1,\dots$

$1,0,1,0,1,0,\dots$

$0,0,1,0,0,1,0,0,1,0,0,1$

$1,1,0,1,1,0,1,1,0,1,1,0$

e) True. by Intermediate value Thm.

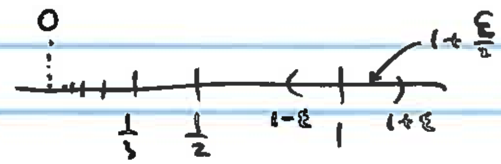
f) True (in $\mathbb{R}$ $\Leftarrow$ by Completeness Axiom.)

2005 (7b) See November 14 notes page 3, Lemma I.

2005 (6)

$$S = \{ \tfrac{1}{n} \mid n \in \mathbb{N} \}$$

(a) Want 1 is not an interior pt



Recall $a \in S$ is called an interior pt of S if $\exists \varepsilon > 0$ s.t. $N(a,\varepsilon) \subseteq S$.

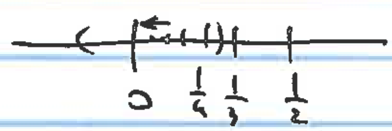
To show: $\forall \varepsilon > 0$ $\quad N(1,\varepsilon) \not\subseteq S$

since $\forall \varepsilon > 0$ $\quad 1 + \frac{\varepsilon}{2} \notin S$

$1 + \frac{\varepsilon}{2} \in N(1,\varepsilon)$; $\quad N(1,\varepsilon) \not\subseteq S$

(b) $0 \in S'$:

recall <u>defn</u> $a \in S' \iff \forall \varepsilon > 0 \quad N^*(a,\varepsilon) \cap S \neq \emptyset$.



$\forall \varepsilon > 0 \quad (-\varepsilon, 0) \cup (0, \varepsilon) = N^*(0,\varepsilon)$

by Arch. principle $\exists\, n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < \varepsilon$

$\forall \varepsilon > 0 \quad N^*(0,\varepsilon) \cap S \neq \emptyset$ since it contains $\frac{1}{n}$ if we choose $n > \frac{1}{\varepsilon}$.

HW 6.2 Exc 8b $f: I \longrightarrow \mathbb{R}$ diff'ble

$f \downarrow \iff f'(x) \leq 0.$

$(\Rightarrow:)$ Assume $f \downarrow$.

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$$

This argument is NOT reversible (next page)

$f\downarrow$ if $x < c$, $f(x) \geq f(c)$

$f(x) - f(c) \geq 0$

$\frac{f(x)-f(c)}{x-c} \leq 0.$

$f \downarrow$ if $x > c$ $f(x) \leq f(c)$

$f(x) - f(c) \leq 0$

$\frac{f(x)-f(c)}{x-c} \leq 0$

≤ 0 in both cases.

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \leq 0$$ since Thm 5.1.8 Cor 4.2.5

$\Leftarrow$: (Uses Mean Value Thm.) Assume $f'(x) \leq 0$

if $x < y$, $x, y \in I$ by M.V.T. $\exists c$ between x & y s.t

$$\frac{f(x)-f(y)}{x-y} = f'(c) \leq 0.$$

$x < y \Rightarrow x - y < 0$

Since $x < y$

$f(x) - f(y) \geq 0 \Rightarrow f(x) \geq f(y)$

$\Rightarrow f$ is decreasing

Caution

Let $s_n \to s$

$(\forall n \ s_n \geq A) \Longrightarrow s \geq A$

$\forall n \ s_n > A \not\Rightarrow s > A$ ex $\frac{1}{n} \to 0$.

$s_n \geq A \not\Leftarrow s \geq A$ ex $-\frac{1}{n} \to \underset{\overset{\shortparallel}{s}}{0} \geq \underset{\overset{\shortparallel}{A}}{0}$

but $-\frac{1}{n} \not\geq \underset{\overset{\shortparallel}{A}}{0}$

Same go with functions:

Let $\lim_{x \to c} g(x) = L$.

$g(x) \geq A \Longrightarrow L \geq A$

$g(x) > A \not\Rightarrow L > A$.

$g(x) \geq A \not\Leftarrow L \geq A$

ex $f(x) = -x^2$

$\lim_{x \to 0} (-x^2) = \underset{\underset{L}{\shortparallel}}{0} \geq \underset{\underset{A}{\shortparallel}}{0}$

but

$-x^2 \not\geq 0$