Oct 31, 2018
7:30pm
①

Review for MT2

Old Tests MT2

2003/#4

$$S = \{ r \in \mathbb{Q} \mid r^2 \le 4 \} = \mathbb{Q} \cap [-2, 2].$$

a) $\text{int } S = \phi$

Proof $x \in \text{int } S \iff \exists \varepsilon > 0 \quad N(x, \varepsilon) \subseteq S.$

But if $\varepsilon > 0 \quad N(x, \varepsilon) = \underbrace{(x-\varepsilon, x+\varepsilon)}_{\text{has } \infty\text{ly many irrational numbers}} \subseteq \mathbb{R}.$

$(x-\varepsilon, x+\varepsilon) \not\subseteq \mathbb{Q}.$

$(x-\varepsilon, x+\varepsilon) \not\subseteq \mathbb{Q} \cap [-2, 2].$

No such x exists. Hence $\text{int } S = \phi.$
$\uparrow$ int S

b) $\sup S = 2 = \max S.$

recall defn of supremum 10/12/18

(i) $\forall x \in S, \quad x \le 2, \quad 2$ is an upper bound.

Least upper bd?

(ii) Let $m' < 2 \quad \exists s = 2 \in S \quad 2 > m'.$

hence $m' < 2 \Rightarrow m'$ is not an upper bd for S.

c) Is S open set? NO (Open $\overset{\text{Prop.}}{\iff} S = \underbrace{\text{int } S}.$)

S is not open $\quad \underset{\phi}{\#} \quad \phi.$

- Is S closed set? NO

$\mathbb{R} - S = (-\infty, -2) \cup (2, \infty) \cup \Big(\underbrace{[-2, 2] \cap \text{Irrationals}}_{\text{int} = \phi} \Big)$

$\mathbb{R} - S$ is not open

S is not closed.

Defn S closed $\iff \text{bd } S \subseteq S.$

OR

- $\text{bd } S = [-2, 2] \not\subseteq [-2, 2] \cap \mathbb{Q} = S \Rightarrow S$ not closed

$$Cl(S) = \overline{S} = S \cup S' = S \cup \text{Bd}\, S = [-2, 2]$$

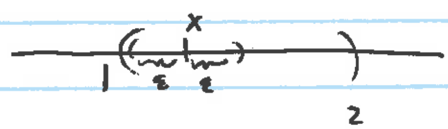
$$\text{bd}\, S = [-2, 2]$$

$$\text{Inf}\, S = -2 = \min S$$

$$\sup S = 2 = \max S.$$

2002 #3 $\quad S = [1, 2)$

(a) Claim $\text{Int}\, S = (1, 2)$



Let $x \in (1, 2)$ be given

$1 < x < 2$

Let $\varepsilon = \min(x-1, 2-x) > 0$

Want To show $(x-\varepsilon, x+\varepsilon) \subseteq [1, 2)$

Recall defn
$x \in \text{int}\, S$:
If $\exists \varepsilon > 0$
$N(x, \varepsilon) \subseteq S$.
$\quad \parallel \qquad\quad \parallel$
$(x-\varepsilon, x+\varepsilon) \subseteq [1, 2)$

Let $y \in (x-\varepsilon, x+\varepsilon)$ be an arbitrary element

$x - \varepsilon < y < x + \varepsilon$

$\varepsilon = \min(x-1, 2-x) \Rightarrow \begin{cases} \varepsilon \leq x - 1 & \Rightarrow \quad 1 \leq x - \varepsilon \\ \varepsilon \leq 2 - x. & \Rightarrow \quad \varepsilon + x \leq 2 \end{cases}$

$1 \leq x - \varepsilon < y < \varepsilon + x \leq 2$

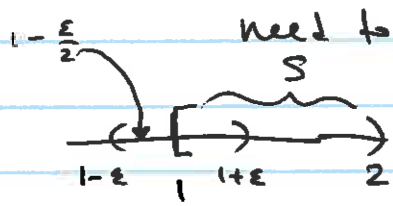
$1 < y < 2.$

This shows $(x-\varepsilon, x+\varepsilon) \subseteq (1, 2) \subseteq [1, 2)$

We obtained $\forall x \in (1, 2)$ x is an interior pt of S $\Big\}$ Hence all of $(1, 2) \subseteq \text{int}\, S. \subseteq S = [1, 2)$

Want $(1, 2) = \text{int}\, S$

need to show $1 \notin \text{int}\, S$



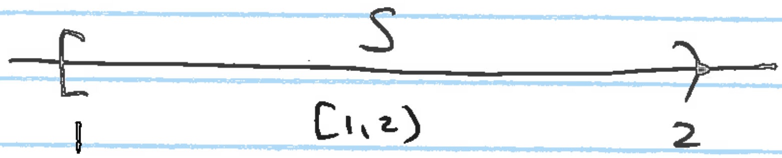
$\forall \varepsilon > 0 \quad 1 - \frac{\varepsilon}{2} \in (1-\varepsilon, 1+\varepsilon)$

$1 - \frac{\varepsilon}{2} \notin S.$

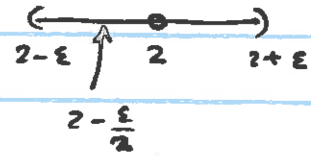
$\forall \varepsilon > 0 \quad N_\varepsilon(1) \nsubseteq S. \quad 1 \notin \text{int}\, S.$

(b) Recall Defn $x \in S'$

$\iff \forall \varepsilon > 0 \quad N^*(x,\varepsilon) \cap S \neq \emptyset$



Let $\varepsilon > 0$ be given. $N^*(2,\varepsilon) = (2-\varepsilon, 2) \cup (2, \varepsilon+\varepsilon)$

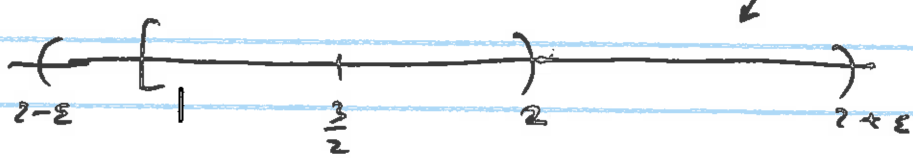


$[1,2) \cap \left((2-\varepsilon, 2) \cup (2, 2+\varepsilon) \right) \neq \emptyset$

since

$2 - \frac{\varepsilon}{2} \in N^*(2,\varepsilon) \cap [1,2) \neq \emptyset$ if $0 < \varepsilon \leq 1$

If $\varepsilon > 1$



$\frac{3}{2} \in N^*(2,\varepsilon) \cap [1,2) \neq \emptyset$

Hence $2 \in S'$.

S open? No

S closed? No

$\bar{S} = [1,2]$.

bd $S = \{1,2\}$

$S' = [1,2]$.

2005 #4 $\quad s_1 = 0$

$s_{n+1} = \sqrt{6 + s_n}\,,\ n \geq 1.$

(a) <u>Claim</u> $\quad 0 \leq S_n \leq 10 \quad p(n)$ Proof by induction.

$0 \leq S_1 = 0 \leq 10$ ✓ $\underline{p(1)}$ Base Case

<u>Want</u> $\quad p(k) \Rightarrow p(k+1)$

$$0 \leq S_k \leq 10 \qquad p(k)$$

$$6 \leq 6 + S_k \leq 16$$

$$0 \leq \sqrt{6} \leq \underbrace{\sqrt{6+S_k}} \leq 4 \leq 10$$

$$0 \leq S_{k+1} \leq 10 \qquad p(k+1)$$

induction step.

This proves claim 1, by induction

<u>Claim 2</u> $\quad S_n \leq S_{n+1} \quad q(n)$ Proof by induction

$0 = S_1 \leq S_2 = \sqrt{6} \quad q(1)$ ✓ Base case

Want $\quad q(k) \Rightarrow q(k+1)$

$$S_k \leq S_{k+1} \qquad q(k)$$

$$0 \leq S_k + 6 \leq S_{k+1} + 6$$

$$S_{k+1} = \sqrt{S_k + 6} \leq \sqrt{S_{k+1} + 6} = S_{k+2}$$

$$S_{k+1} \leq S_{k+2} \qquad q(k+1)$$

induction step.

this proves claim 2.

(b) By Monotone Convergence Thm:

"Every bounded monotone sequence in $\mathbb{R}$ converges."

So, $\exists \lim s_n = L \in \mathbb{R}$.

$$0 \leq L = \lim \underbrace{S_{n+1}}_{\geq 0} = \lim \sqrt{6 + S_k} = \sqrt{6 + L}$$

$L = \sqrt{6+L}$

$L^2 - L - 6 = 0$

$(L-3)(L+2) = 0$

$\Rightarrow L = 3.$

2005 #4

c) $s_1 = k$ given

If $s_1 < -6$, no sequence exists.

If $-6 \le s_1 < 3$ $s_n \nearrow 3$.

If $s_1 = 3$, then $s_n = 3 \to 3$

If $s_1 > 3$ $s_n \searrow 3$.

2005 #2 By using $\varepsilon - N$ defn only prove $\lim_{n\to\infty} \frac{n+3}{n^2-2} = 0$.

Want to show

$\forall \varepsilon > 0 \ \exists N \text{ then } \in \mathbb{N} \left(n \ge N \Rightarrow \left| \frac{n+3}{n^2-2} - 0 \right| < \varepsilon \right.$

Scrapwork:

$$\left| \frac{n+3}{n^2-2} - 0 \right|$$

$$\frac{n+3}{n^2-2} \sim \frac{n}{n^2} = \frac{1}{n} \quad \text{for large } n.$$

Can I prove $\left| \frac{n+3}{n^2-2} \right| < \frac{2}{n}$?

$\left(\frac{1}{n} \text{ will not work} \right)$

$\frac{n+3}{n^2-2} < \frac{2}{n}$?

$n^2 + 3n < 2(n^2-2)$?

$n^2 + 3n < 2n^2 - 4$?

$0 < n^2 - 3n - 4$? if

$0 < (n-4)(n+1)$ Yes $n \ge 5$

PTO for actual proof

(a) Proof of $\lim \frac{n+3}{n^2-2} = 0$.

$\forall n \in \mathbb{N}$, $n \geq 5$, we have

$$0 < (n-4)(n+1)$$
$$0 < n^2 - 3n - 4$$
$$n^2 + 3n < 2n^2 - 4$$
$$n(n+3) < 2(n^2-2)$$
$$\frac{n+3}{n^2-2} < \frac{2}{n} \quad \text{if } n \geq 5, \; n^2-2>0$$

Let $\varepsilon > 0$ be given

Choose $N \in \mathbb{N}$, $N > \max(\frac{2}{\varepsilon}, 5)$

$\forall n \geq N$

$$|s_n - L| = \left|\frac{n+3}{n^2-2} - 0\right| = \frac{n+3}{n^2-2} < \frac{2}{n} \leq \underbrace{\frac{2}{N} < \varepsilon}_{\frac{2}{\varepsilon} < N}.$$

2006 #2(a) $\lim_{n\to\infty} \frac{6n}{3n+1} = \lim_{n\to\infty} \frac{n \cdot 6}{n(3+\frac{1}{n})} = \lim_{n\to\infty} \frac{6}{3+\frac{1}{n}} = 2$

By using Limit Thms.

since

$\lim_{n\to\infty} 6 = 6$

$\lim_{n\to\infty} 3 + \frac{1}{n} = 3$ since $\lim \frac{1}{n} = 0$.

$\lim \frac{s_n}{t_n} = \frac{\lim s_n}{\lim t_n}$ if both $(s_n), (t_n)$ are convergent

$t_n \neq 0$

$\lim t_n = t \neq 0$

(b) $\varepsilon - N$ proof (No limit thms.)

2004 #1. Prove

$\forall c>0 \quad (a_n \to +\infty \implies ca_n \to +\infty)$.

Recall defn $s_n \to \infty$ if

$$\forall M\ \exists N\ \forall n \in \mathbb{N}\ (n \geq N \Rightarrow s_n > M)$$

Proof $a_n \to +\infty$ Given

Given M $\exists N$ $\forall n \geq N$ $a_n > M$.

Given M', take $M = \frac{M'}{c}$ $\exists N$ $\forall n \geq N$ $a_n > M = \frac{M'}{c}$

Take $N' = N$ $\qquad ca_n > M'$.

Hence Given M' $\exists N'$ $\forall n \geq N'$ $ca_n > M'$

$ca_n \to \infty$ conclusion.

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