Review Session

Sept 26, 2018

1b(2006) $\frac{u-2}{u} \geq 2 \implies u < 0$

Proof by contradiction

Suppose $u \geq 0$ and $\frac{u-2}{u} \geq 2$.

$u = 0$ is not possible since $\frac{u-2}{u}$ is defined.

$$\frac{u-2}{u} \geq 2$$

$$(u \geq 0) \implies u \cdot \frac{u-2}{u} \geq 2u$$

$$u - 2 \geq 2u.$$

$$\left.\begin{array}{r} -2 \geq u \\ u \geq 0 \end{array}\right\} \text{ no such } u. \text{ Contradiction.}$$

OR $\frac{u-2}{u} \geq 2$

$$1 - \frac{2}{u} \geq 2$$

$-1 \geq \frac{2}{u}$. can't be true if $u > 0$

$u = 0$ is not possible either

So $u < 0$.

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Exc 2.1.19 p 51

<u>Prove</u> If $U = A \cup B$ & $A \cap B = \emptyset$, then $A = U \setminus B$.

<u>Proof</u> $U = A \cup B$, $\emptyset = A \cap B$ } given

<u>To show</u> (i) $A \subseteq U - B$

(ii) $U \setminus B \subseteq A$.

To show (i): if $A = \emptyset$, then $\emptyset \subseteq U \setminus B$. done.

If $A \neq \emptyset$, then let $a \in A$ be any elt.

$U = A \cup B$

$a \in A \Rightarrow (a \in A$ or $a \in B)$

$\Rightarrow a \in A \cup B = U$.

$a \in U$

Suppose $a \in B$, but then

$a \in A \Rightarrow a \in A \cap B = \emptyset$.

no such a exists

Conclusion $a \notin B$.

We showed $a \in A \Rightarrow a \in U$ & $a \notin B$

$\Rightarrow a \in U \setminus B$.

$A \subseteq U \setminus B$.

To show (ii) $U \setminus B \subseteq A$.

if $U \setminus B = \emptyset$, there is nothing to prove: $\emptyset \subseteq A$

If $U - B \neq \emptyset$, choose any $a \in U \setminus B$.

$U = A \cup B$

$a \in U \setminus B \Rightarrow a \in U$ & $a \notin B$

$\Rightarrow a \in (A \cup B)$ and $a \notin B \Rightarrow (a \in A$ or $a \in B)$ & $a \notin B$ $\Rightarrow a \in A$.

p93 Exc 2.4.3 d

find a bijection

$[0,1) \longrightarrow (0,1)$

$0 \longrightarrow \frac{1}{2}$

$\frac{1}{2} \longrightarrow \frac{1}{3}$

$\frac{1}{3} \longrightarrow \frac{1}{4}$

$\vdots$

$\frac{1}{n} \longrightarrow \frac{1}{n+1}$

$$\ell(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}. \\ x & \text{if } x \neq 0, \frac{1}{n} \text{ for all } n \in \mathbb{N} \end{cases}$$

Exc 2.1 22 $\quad A \setminus (B \setminus A) \neq B \setminus (A \setminus B)$ in general

Example $\quad A = \{1\}$

$B = \{0\}$

$B \setminus A = \{0\}$

$A \setminus (B \setminus A) = \{1\} - \{0\} = \{1\}$

$A \setminus B = \{1\} \setminus \{0\} = \{1\}$

$B \setminus (A \setminus B) = \{0\} - \{1\}$
$= \{0\}$

$A \setminus (B \setminus A) = \{1\} \neq \{0\} = B \setminus (A \setminus B).$

How do we show that $A \setminus (B \setminus A) = A$?

① Let $a \in A \setminus (B \setminus A) \Longrightarrow a \in A$ and $a \notin B \setminus A$

$\Longrightarrow a \in A$

$A \setminus (B \setminus A) \subseteq A$.

② Let $a \in A$, $a \notin B \setminus A$ (since $a \in A$) explain

$a \in A \setminus (B \setminus A)$

$A \subseteq A \setminus (B \setminus A)$

If $a \in B \setminus A$ were true, then $a \in B$, $a \notin A$ but I know $a \in A$ so $a \in B \setminus A$ is false.

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old Exam 2006 #4 a) Assume $|x| + |y| \geq |x+y|$ (*)

To show $\underbrace{|x_1 + x_2 + \dots + x_n| \leq |x_1| + |x_2| + \dots + |x_n|}_{P(n)}$.

$P(2)$ is true, (*) $|x_1 + x_2| \leq |x_1| + |x_2|$

Want to show $\forall k \geq 2 \ (p(k) \Longrightarrow p(k+1))$

$p(k)$ $\quad |x_1 + \dots + x_k| \leq |x_1| + |x_2| + \dots + |x_k|$.

$|x_1 + x_2 + \dots + x_{k+1}| = |\underbrace{(x_1 + \dots + x_k)}_{x} + \underbrace{(x_{k+1})}_{y}|$

$\underset{p(2)}{\leq} |x_1 + \dots + x_k| + |x_{k+1}| \underset{p(k)}{\leq} |x_1| + \dots + |x_k| + |x_{k+1}|$

Hence $p(k+1)$ is true

True/False

2006) 5 (a) false

$$\sim (x+y>1 \Rightarrow 0 \le xy \le 1)$$

$x+y>1$ and not $(0 \le xy$ and $xy \le 1)$

$x+y>1$ and $(0 > xy$ or $xy > 1)$

$\equiv$ no   no

original $\exists x \forall y \; (x+y) \le 1$ OR ...

True (b) Both are true (i) $\forall x \forall y \; \exists z = y-x$ s.t. $x+z=y$.

(ii) $\forall x \forall y \; \exists z = y-x+1$ so that $x+z = y+1 \ne y$.

False (c) $X = \{ \sqrt{\frac{p}{q}} \mid \frac{p}{q} \in \mathbb{Q}, \frac{p}{q} > 0 \}$

bijection $\quad f(\frac{p}{q}) = \sqrt{\frac{p}{q}}$.

$\{ \frac{p}{q} \mid \frac{p}{q} \in \mathbb{Q}, \frac{p}{q} \}$ countable $(\subseteq \mathbb{Q}.)$

$\Rightarrow X$ countable

see class notes about $\mathbb{Q}$ being countable.

(d) N/A

(e) False:

$$\sim \forall n \in A \; p(n) \equiv \exists n \in A \sim p(n)$$

need to find <u>one</u> false $p(n)$.

Old Exam 2004 #1

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend left, "g \circ f"] & B \arrow[r, "g"] & C \end{tikzcd}$$

(a)

Show: f & g are injective $\Longrightarrow$ $g \circ f$ is injective.

recall defn of injective

$\forall a_1, a_2 \in A \,(a_1 \neq a_2 \Longrightarrow f(a_1) \neq f(a_2))$

OR

given: f inj, g inj

$\forall a_1, a_2 \in A \,(f(a_1) = f(a_2) \Longrightarrow a_1 = a_2)$. ✓ (1)

$\forall b_1, b_2 \in B \,(g(b_1) = g(b_2) \Longrightarrow b_1 = b_2)$. ✓ (2)

Want: $g \circ f$ injective

Want: $\forall a_1, a_2 \in A \quad (g \circ f)(a_1) = (g \circ f)(a_2)$

$\Longrightarrow a_1 = a_2$.

Let a_1, a_2 be elements in A s.t

$g(f(a_1)) = (g \circ f)(a_1) = (g \circ f)(a_2) = g(f(a_2))$

Let $b_1 = f(a_1)$ Let $b_2 = f(a_2)$

$g(b_1) = g(b_2) \Longrightarrow b_1 = b_2$ by (2)

$b_1 = f(a_1) = b_2 = f(a_2)$

$f(a_1) = f(a_2) \Longrightarrow a_1 = a_2$ by (1).

Old Exams 2004) 1b f, g surjective $\Rightarrow g \circ f$ is surjective

Recall defn of surjective $f : A \to B$

$\forall b \in B \ \exists a \in A \ f(a) = b$. (1)

surjective $g : B \to C$

$\forall c \in C \ \exists b \in B \ g(b) = c$. (2)

Want: $g \circ f : A \to C$ surjective

Want: $\forall c \in C \ \exists \ a \in A$ s.t. $g \circ f(a) = c$.

Given $c \in C$, find $b \in B$ s.t. $g(b) = c$ by (2)

For the b you have, find an $a \in A$ $f(a) = b$

$(g \circ f)(a) = g(f(a)) = g(b) = c$

$$A \xrightarrow{f} B \xrightarrow{g} C$$

$$a \longrightarrow b \longrightarrow c$$

c) f, g bijective (bijective $\overset{\text{defn}}{\Longleftrightarrow}$ injective $\times$ surjective)

$\Rightarrow f \times g$ injective

$\Rightarrow g \circ f$ injective by part (a)

f, g bijective

$\Rightarrow f \times g$ surjective

$\Rightarrow g \circ f$ surjective by part (b)

$\Rightarrow g \circ f$ is bijective

$\Rightarrow (g \circ f)^{-1}$ exists by defn 2.3.23.

2004 True/False

(d) CAUTION (Compare ① & ② & exam question.)

TRUE ① There are sets A, B, C s.t. $(A-B)\cup C = (A\cup C)-B$

$A=\{1\}$, $B=\{2\}$, $C=\{3\}$

$(A-B)\cup C = \{1,3\} = (A\cup C)-B$

FALSE ② $\forall$ sets A, B, C $(A-B)\cup C = (A\cup C)-B$.

Take any $A=B=C\neq\emptyset$.

$(A-B)\cup C = \emptyset\cup C = C$

$(A\cup C)-B = A-B = \emptyset$ $\neq$

FALSE (c) $\forall x,y\in\mathbb{R}, \exists n\in\mathbb{N}$ $nx>y$

$x=0, y=1$ $\exists$ no n $n\cdot 0>1$.

Arch Principle $\forall x,y>0, \exists n\in\mathbb{N}, nx>y$

FALSE (b) $\mathbb{Q}$ is countable

$\mathbb{R}\setminus\mathbb{Q}$ is uncountable

can't be equinumerous.

FALSE (a) $\forall x\in\mathbb{R}\ \exists y\in\mathbb{R}\ (xy>1 \Rightarrow x>y)$

$\downarrow$ negate

$\exists x\in\mathbb{R}\ \forall y\in\mathbb{R}\ (xy>1$ and $x\leq y)$

But Given $\exists x\in\mathbb{R}\ \forall y\in\mathbb{R}\ (x\leq y \Rightarrow xy\leq 1)$

Caution $p\Rightarrow q$ is equivalent to $\sim q\Rightarrow\sim p$

$p\Rightarrow q$ is the opposite/negation of $p\wedge\sim q$.