Dec 7 2018

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In class Review 205 MLH

Dec 11 Tuesday 3–5pm

Office Hours Wed 10–12

Thu 10–12 B20F MLH

Posted on the course webpages

- Review File
- Old final Exams.

Proofs from 4.4, 5.1, 5.2, 5.3, 6.1, 6.2, Compactness

[No proofs from chap 7

Only applications, HW, Examples from chap 7

Of course, HW, applications, and examples can be from chaps 4–6 as well

7.3

FUNDAMENTAL THMS of CALCULUS

FTC I Let $f:[a,b]\to$ be integrable (bdd by defn.)

Define
$$F(x) = \int_a^x f(t)\,dt \quad : [a,b] \to \mathbb{R}.$$

Then a) F is uniformly continuous

b) If f is continuous, then F is diff'ble and $F' = f$.

FTC II If $F:[a,b]\to\mathbb{R}$ is diff'ble on $[a,b]$ and if F' is integrable on $[a,b]$,

then
$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Example ①
$$\lim_{x\to 2} \frac{1}{x-2}\int_2^x \cos\sqrt{t}\,dt$$

Let $F(x) = \int_2^x \cos\sqrt{t}\,dt$

$$\lim_{x\to 2}\frac{1}{x-2}\int_2^x \cos\sqrt{t}\,dt = \lim_{x\to 2}\frac{F(x)-F(2)}{x-2} = F'(2) = f(2)$$
$$= \cos\sqrt{2}$$

Example ②
$$\frac{d}{dx}\int_{x^2}^{6x} e^{t^2}\,dt.$$

Let $F(x) = \int_0^x e^{t^2}\,dt$ so that $F'(x) = e^{x^2}$

$$\int_{a(x)}^{b(x)} F'(x)\,dx = F(b(x)) - F(a(x))$$

$$\frac{d}{dx}\int_{a(x)}^{b(x)} F'(x)\,dx = \frac{d}{dx}\Big(F(b(x)) - F(a(x))\Big)$$

1 $$= F'(b(x))\cdot b'(x) - F'(a(x))\cdot b'(x)$$

$$\frac{d}{dx}\int_{x^2}^{6x} e^{t^2}\,dt = e^{(6x)^2}\cdot 6 - e^{(x^2)^2}\cdot 2x = 6e^{36x^2} - 2x\,e^{x^4}.$$

$$F'(x) = e^{x^2}$$

Example 3 Exc. 7. p 299

$$F(x) = \int_0^x x e^{t^2}\,dt \qquad 0 \le x \le 1.$$

$$F(x) = x\int_0^x e^{t^2}\,dt$$

$$F'(x) = 1\cdot\int_0^x e^{t^2}\,dt + x\cdot e^{x^2}$$

$$F''(x) = e^{x^2} + 1\cdot e^{x^2} + x\cdot 2x e^{x^2}$$

$$= e^{x^2}(2 + 2x^2)$$

Proof of FTC I

(a) Let f be integrable and hence bounded.
$\exists B \in \mathbb{R} \quad \forall x \in [a,b] \quad |f(x)| \le B.$

for $y \geq x$
$$|F(y)-F(x)| = \left| \int_a^y f(t)dt - \int_a^x f(t)dt \right|$$
$$= \left| \int_x^y f(t)dt \right| \le \int_x^y |f(t)|dt \le \int_x^y B dt = B(y-x)$$
$$= B|y-x|.$$

for $x \geq y$
$$|F(y)-F(x)| = |F(x)-F(y)| \le B|x-y| = B|y-x|.$$

If $B>0$, $\forall \varepsilon > 0 \ \exists \ \delta = \frac{\varepsilon}{B} \quad \forall x,y \in [a,b]$
$$|x-y| < \delta \implies |F(x)-F(y)| < B|x-y| < B \cdot \delta = \varepsilon.$$

If $B=0$, then $f \equiv 0$, $F(x) \equiv 0$.
Hence F is uniformly continuous

FTC I (b) $f:[a,b] \to \mathbb{R}$ continuous on $[a,b]$

$\Longrightarrow$ f is integrable, $F(x) = \int_a^x f(t)$ is diff'ble on $[a,b]$

and $F'(x) = f(x)$.

Let c be fixed, $c \in [a,b]$.

Proof Let $\varepsilon > 0$ be given

$\exists\, \delta > 0 \quad \forall x \in [a,b], \quad |x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$

If $0 < |x-c| < \delta$

$$\left| \frac{F(x) - F(c)}{x-c} - f(c) \right| = \left| \left(\frac{1}{x-c} \int_c^x f(t)\,dt \right) - f(c) \right|$$

$$= \left| \frac{1}{x-c} \int_c^x f(t)\,dt - \frac{1}{x-c} \int_c^x f(c)\,dt \right|$$

$$= \left| \frac{1}{x-c} \right| \left| \int_c^x f(t) - f(c)\,dt \right|$$

$$\leq \left| \frac{1}{x-c} \right| \int_c^x |f(t) - f(c)|\,dt$$

$$\leq \frac{1}{x-c} \int_c^x \varepsilon\,dt = \varepsilon$$

We showed that $\forall \varepsilon > 0\ \exists\, \delta > 0$ s.t.

$$0 < |x-c| < \delta \Rightarrow \left| \frac{F(x) - F(c)}{x-c} - f(c) \right| < \varepsilon$$

$$\lim_{x \to c} \frac{F(x) - F(c)}{x-c} = f(c).$$

$$F'(c) = f(c).$$

(Not in the test, since this requires multivariable calculus.)

Example 4

Find $\frac{d}{dx}\int_{x^3}^{x^2} \frac{1}{t} \sin x t^2 \, dt$ for $x>0$

For $a,b \in \mathbb{R}$ (Leibnitz Rule) $\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial f}{\partial x}(x,t)\,dt$ for f diff'ble in (x,t).

What do we do $\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt$?

Let $F(x,t) = \int_{a_0}^{t} f(x,u)\,du$ (partial integration)

$$\frac{\partial F}{\partial t} = f(x,t)$$

$$\frac{\partial F}{\partial x} = \int_{a_0}^{t} \frac{\partial f}{\partial x}(x,u)\,du.$$

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t) = \frac{d}{dx}\Big(F(x,b(x)) - F(x,a(x))\Big)$$

$$= F_x(x,b(x)) + F_t(x,b(x))\,b'(x) - F_x(x,a(x)) - F_t(x,a(x))\,a'(x)$$

$$= \int_{a_0}^{b(x)} \frac{\partial f}{\partial x}(x,u)\,du - \int_{a_0}^{a(x)} \frac{\partial f}{\partial x}(x,u)\,du +$$

$$+ f(x,b(x))\,b'(x) - f(x,a(x))\,a'(x)$$

$$= \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}(x,u)\,du + f(x,b(x))\,b'(x) - f(x,a(x))\cdot a'(x)$$

$$f(x,t) = \frac{1}{t} \sin x t^2$$

$$\frac{d}{dx} \int_{x^3}^{x^2} \underbrace{\frac{1}{t} \sin x t^2}_{\swarrow \frac{\partial}{\partial x}} \, dt \qquad \text{for} \quad x > 0$$

$$= \int_{x^3}^{x^2} t \cos x t^2 \, dt + \Big(\underbrace{\frac{1}{x^2} \sin x^5}_{f(x,x^2)}\Big) \cdot 2x - \Big(\underbrace{\frac{1}{x^3} \sin x^7}_{f(x,x^3)}\Big) 3x^2$$

$(x \text{ fixed})$ $\quad u = xt^2$, $du = 2xt$

$$= \frac{1}{2x} \sin x t^2 \Big|_{x^3}^{x^2} + \frac{2}{x} \sin x^5 - \frac{3}{x} \sin x^7$$

$$= \frac{1}{2x} \sin x^5 - \frac{1}{2x} \sin x^7 + \frac{2}{x} \sin x^5 - \frac{3}{x} \sin x^7$$

$$= \frac{5}{2x} \sin x^5 - \frac{7}{2x} \sin x^7$$