Dec 5, 2018

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Review : Tue 3-5 Room TBA

Office Hrs : W: 10-12, Th: 10-12 } B20F MLH.

7.1 + 7.3 Fund Thms Calculus + Applications

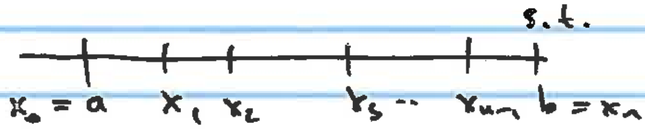
- No proofs from 7.1, 7.3 in final
- Yes: Applications, statements, T/F, long.
- Concentrate on Class notes + HW

Review Defn of Integration

7.1

Defn Let $[a,b] \subseteq \mathbb{R}$,

- A partition P of $[a,b]$, $P = \{x_0, x_1, \dots x_n\}$ is s.t. $a = x_0 < x_1 < \dots x_{n-1} < x_n = b$

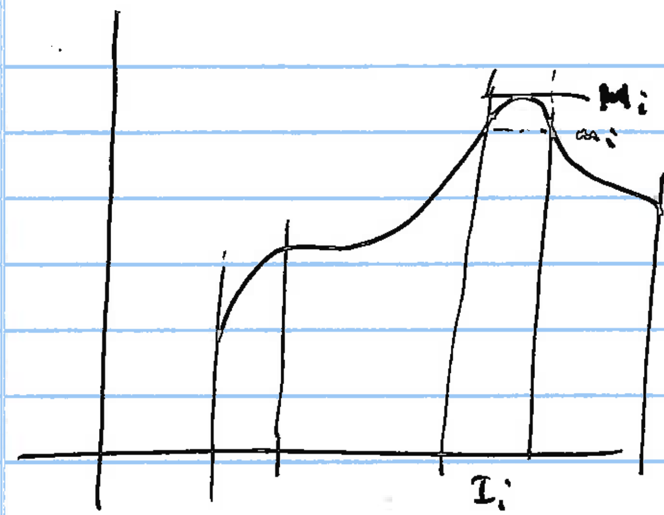


- A partition Q is called a refinement of P (a partition) if $P \subseteq Q$.

Defn Let $f:[a,b] \to \mathbb{R}$ be a bounded function.
Let P be a partition of $[a,b]$, $P =$
Set $\Delta x_i = x_i - x_{i-1}$

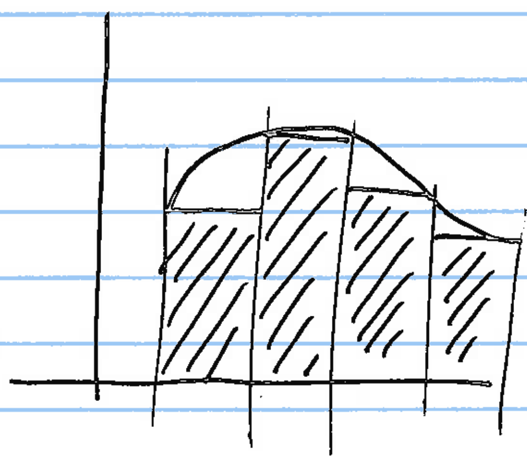
$M_i = \sup\{f(x) \mid x_{i-1} \le x \le x_i\}$ $\quad I_i = [x_{i-1}, x_i]$.

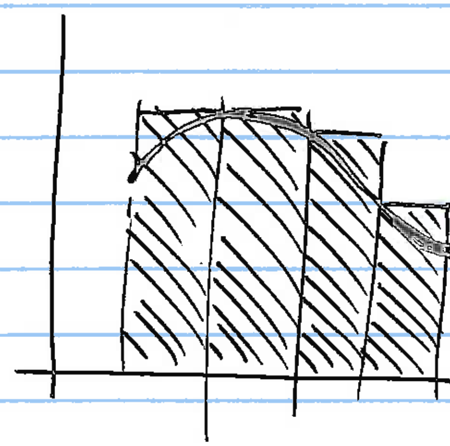
$m_i = \inf\{f(x) \mid x_{i-1} \le x \le x_i\}$



Define $U(f,P) = \sum_{i=0}^{n} M_i \, \Delta x_i$

$L(f,P) = \sum_{i=0}^{n} m_i \, \Delta x_i$





$$L(f,P) \underset{\text{always.}}{\leq} U(f,P)$$

Prop If Q is a refinement of P; $P \subseteq Q$ then

$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$$

$$\xrightarrow[\text{refining}]{} \cdots \qquad \cdots \xleftarrow[\text{refining}]{}$$

Defn

$$L(f) = \sup \{ L(f,P) \mid P \text{ is any partition of } [a,b] \}$$

$$U(f) = \inf \{ U(f,P) \mid \quad \text{"} \quad \}$$

So:

If $P \subseteq Q$, f is bounded:

$$L(f,P) \le L(f,Q) \le L(f) \le U(f) \le U(f,Q) \le U(f,P)$$

sup ⟶ ⟵ inf

Is it $=$?

Defn Let $f : [a,b] \to \mathbb{R}$ be bounded.
f is called integrable if $L(f) = U(f)$.
If f is integrable then

$$\int_a^b f(x)\,dx = L(f) = U(f).$$

Ex

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} - \mathbb{Q}. \end{cases} : [0,1] \to \{0,1\}$$

$\forall\ [a,b] \subseteq [0,1],\ a < b$

$$\left.\begin{array}{l} \inf\limits_{[a,b]} f = 0 \\ \sup\limits_{[a,b]} f = 1 \end{array}\right\} \begin{array}{l} \text{Density of } \mathbb{Q} \text{ in } \mathbb{R}. \\ \text{Density of } \mathbb{R} - \mathbb{Q} \text{ in } \mathbb{R}. \end{array}$$

$U(f) = 1$

$L(f) = 0$

f is not integrable

7.2.2 <u>Prop</u> (i) f continuous on $[a,b] \Rightarrow f$ is integrable on $[a,b]$.

(ii) f has finitely many jump discontinuities (and continuous elsewhere) on $[a,b]$
$\Rightarrow f$ is integrable on $[a,b]$.

How does one prove Prop 7.2.2?

<u>Defn</u> A function $f: D \subseteq \mathbb{R} \to \mathbb{R}$ is called uniformly continuous if
$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x, y \in D \ (|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon)$$
$\delta = \delta(\varepsilon)$, but independent of x & y

- Compare Continuous on D — independent of y but depends on x & ε.
$$\forall x \in D \ \forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon, x) > 0 \ \forall y \in D \ (|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon)$$

- <u>Lemma</u>: uniformly continuous $\Rightarrow$ continuous

- $f(x) = \frac{1}{x} : (0,1) \to (1, \infty)$ is continuous but <u>not</u> uniformly continuous

<u>Theorem</u>: Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$ be continuous and let D be compact.
Then f is uniformly continuous.

<u>Proof</u> of $\left. \begin{array}{l} f : [a,b] \to \mathbb{R} \\ \text{continuous} \end{array} \right\} \Longrightarrow$ f is integrable over $[a,b]$.
where $a<b$

i.e. $\int_a^b f(x)dx$ exists.

f continuous on $[a,b] \Longrightarrow f$ is unif continuous (by Thm).

Let $\varepsilon > 0$ be given.

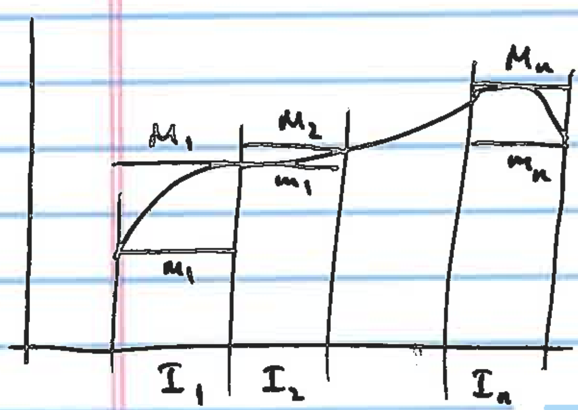
$\exists\, \delta > 0$ s.t.

$$\forall x,y \in [a,b] \; \left(|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a} \right.$$

$\exists\, n \in \mathbb{N}$ s.t. $\frac{b-a}{n} < \delta$.

Take the partition P of $[a,b]$ by dividing into n equal pieces.

$$x_i = a + \left(\frac{b-a}{n}\right) i \qquad 0 \le i \le n; \quad \Delta x_i = \frac{b-a}{n}$$



$I_i = [x_{i-1}, x_i]$

$m_i = \inf_{I_i} f = \min_{I_i} f$

$M_i = \sup_{I_i} f = \max_{I_i} f$

$\forall x,y \quad x,y \in I_i \Longrightarrow |x-y| < \delta$

$\Longrightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a}$.

$\Longrightarrow 0 \le M_i - m_i < \frac{\varepsilon}{b-a}$.

$$0 \le U(f,P) - L(f,P) = \sum_{i=1}^{n} M_i \Delta x_i - \sum_{i=1}^{n} m_i \Delta x_i$$

$$= \sum_{i=1}^{n} (M_i - m_i) \Delta x_i < \sum_{i=1}^{n} \frac{\varepsilon}{b-a} \cdot \frac{b-a}{n} = \varepsilon.$$

$\forall \varepsilon > 0 \ \exists P$ s.t. $\quad U(f,P) - L(f,P) < \varepsilon$

$$L(f,P) \le L(f) \le U(f) \le U(f,P)$$

differ by at most ε.

$\Longrightarrow \quad |U(f) - L(f)| < \varepsilon \quad \forall \varepsilon.$

$\Longrightarrow \quad U(f) = L(f)$

$\Longrightarrow \quad \int_a^b f(x)\,dx$ exists & equals $U(f) = L(f)$.